Contents

CHAPTER 1

Introduction to Calculus

The right way to begin a calculus book is with calculus. This chapter will jump directly into the two problems that the subject was invented to solve. You will see what the questions are, and you will see an important part of the answer. There are plenty of good things left for the other chapters, so why not get started?

The book begins with an example that is familiar to everybody who drives a car. It is calculus in action—the driver sees it happening. The example is the relation between the *speedometer* and the *odometer*. One measures the speed (or *velocity*); the other measures the *distance traveled*. We will write v for the velocity, and f for how far the car has gone. The two instruments sit together on the dashboard:
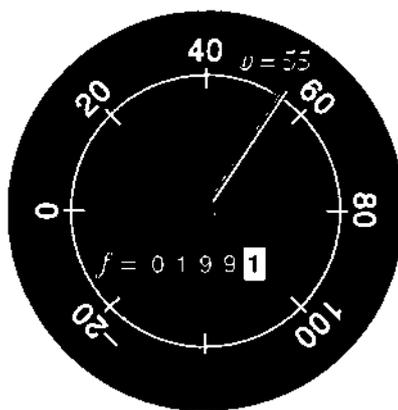
Fig. 1.1   Velocity v and total distance f (at one instant of time).

Notice that the units of measurement are different for v and f. The distance f is measured in kilometers or miles (it is easier to say miles). The velocity v is measured in km/hr or *miles per hour*. A unit of *time* enters the velocity but not the distance. Every formula to compute v from f will have f divided by time.

The central question of calculus is the relation between v and f.

1

Can you find v if you know f, and vice versa, and how? If we know the velocity over the whole history of the car, we should be able to compute the total distance traveled. In other words, if the speedometer record is complete but the odometer is missing, its information could be recovered. One way to do it (without calculus) is to put in a new odometer and drive the car all over again at the right speeds. That seems like a hard way; calculus may be easier. But the point is that *the information is there.* If we know everything about v, there must be a method to find f.

What happens in the opposite direction, when f is known? If you have a complete record of distance, could you recover the complete velocity? In principle you could drive the car, repeat the history, and read off the speed. Again there must be a better way.

The whole subject of calculus is built on the relation between v and f. The question we are raising here is not some kind of joke, after which the book will get serious and the mathematics will get started. On the contrary, *I am serious now*—and the mathematics has already started. We need to know how to find the velocity from a record of the distance. (That is called **differentiation**, and it is the central idea of **differential calculus**.) We also want to compute the distance from a history of the velocity. (That is **integration**, and it is the goal of **integral calculus**.)

Differentiation goes from f to v; integration goes from v to f. We look first at examples in which these pairs can be computed and understood.

CONSTANT VELOCITY

Suppose the velocity is fixed at $v = 60$ (miles per hour). Then f increases at this constant rate. After two hours the distance is $f = 120$ (miles). After four hours $f = 240$ and after t hours $f = 60t$. We say that f increases **linearly** with time—its graph is a straight line.
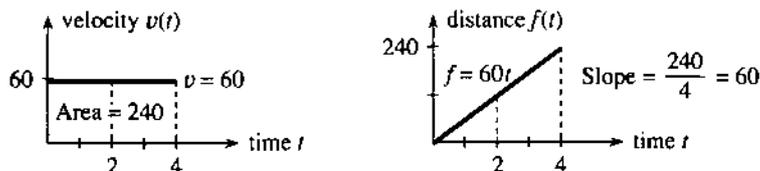
Fig. 1.2   Constant velocity $v = 60$ and linearly increasing distance $f = 60t$.

Notice that this example starts the car at full velocity. No time is spent picking up speed. (The velocity is a "step function.") Notice also that the distance starts at zero; the car is new. Those decisions make the graphs of v and f as neat as possible. One is the horizontal line $v = 60$. The other is the sloping line $f = 60t$. This v, f, t relation needs algebra but not calculus:

$$\text{if } v \text{ is constant and } f \text{ starts at zero then } f = vt.$$

The opposite is also true. When f increases linearly, v is constant. *The division by time gives the slope.* The distance is $f_1 = 120$ miles when the time is $t_1 = 2$ hours. Later $f_2 = 240$ at $t_2 = 4$. At both points, the ratio f/t is 60 miles/hour. Geometrically, *the velocity is the slope of the distance graph*:

$$\text{slope} = \frac{\text{change in distance}}{\text{change in time}} = \frac{vt}{t} = v.$$
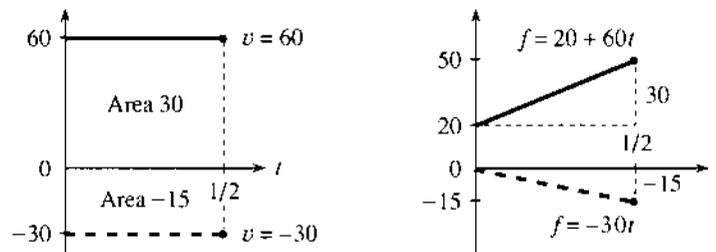
Fig. 1.3 Straight lines $f = 20 + 60t$ (slope 60) and $f = -30t$ (slope -30).

The slope of the f-graph gives the v-graph. Figure 1.3 shows two more possibilities:

1. The distance starts at 20 instead of 0. The distance formula changes from $60t$ to $20 + 60t$. The number 20 cancels when we compute *change* in distance—so the slope is still 60.
2. When v is *negative*, the graph of f goes *downward*. The car goes backward and the slope of $f = -30t$ is $v = -30$.

I don't think speedometers go below zero. But driving backwards, it's not that safe to watch. If you go fast enough, Toyota says they measure "absolute values"—the speedometer reads $+30$ when the velocity is -30. For the odometer, as far as I know it just stops. It should go backward.†

VELOCITY *vs.* DISTANCE: SLOPE *vs.* AREA

How do you compute f from v? The point of the question is to see $f = vt$ *on the graphs.* We want to start with the graph of v and discover the graph of f. Amazingly, the opposite of slope is *area.*

The distance f is the area under the v-graph. When v is constant, the region under the graph is a rectangle. Its height is v, its width is t, and its area is v times t. This is *integration,* to go from v to f by computing the area. We are glimpsing two of the central facts of calculus.

> **1A** *The slope of the f-graph gives the velocity v. The area under the v-graph gives the distance f.*

That is certainly not obvious, and I hesitated a long time before I wrote it down in this first section. The best way to understand it is to look first at more examples. The whole point of calculus is to deal with velocities that are *not* constant, and from now on v has several values.

EXAMPLE (*Forward and back*) There is a motion that you will understand right away. The car goes forward with velocity V, and comes back at the same speed. To say it more correctly, the *velocity in the second part is* $-V$. If the forward part lasts until $t = 3$, and the backward part continues to $t = 6$, *the car will come back where it started.* The total distance after both parts will be $f = 0$.

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†This actually happened in *Ferris Bueller's Day Off*, when the hero borrowed his father's sports car and ran up the mileage. At home he raised the car and drove in reverse. I forget if it worked.
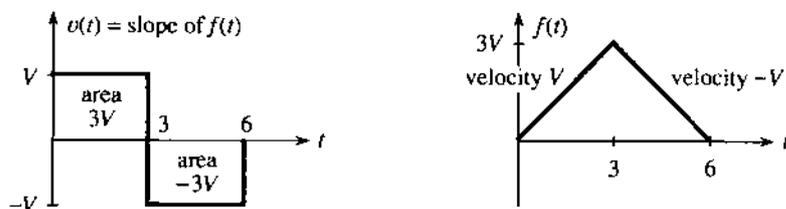
Fig. 1.4   Velocities $+V$ and $-V$ give motion forward and back, ending at $f(6) = 0$.

The v-graph shows velocities $+V$ and $-V$. The distance starts up with slope $+V$ and reaches $f = 3V$. Then the car starts backward. The distance goes down with slope $-V$ and returns to $f = 0$ at $t = 6$.

Notice what that means. The total area "under" the v-graph is zero! A negative velocity makes the distance graph go *downward* (*negative slope*). The car is moving backward. *Area below the axis in the v-graph is counted as negative.*

FUNCTIONS

This forward-back example gives practice with a crucially important idea—the concept of a "*function*." We seize this golden opportunity to explain functions:

The number $v(t)$ is the value of the function v at the time t.

The time t is the **input** to the function. The velocity $v(t)$ at that time is the **output**. Most people say "v of t" when they read $v(t)$. The number "v of 2" is the velocity when $t = 2$. The forward-back example has $v(2) = +V$ and $v(4) = -V$. The function contains the whole history, like a memory bank that has a record of v at each t.

It is simple to convert forward-back motion into a formula. Here is $v(t)$:

$$v(t) = \begin{cases} +V & \text{if} \quad 0 < t < 3 \\ ? & \text{if} \quad t = 3 \\ -V & \text{if} \quad 3 < t < 6 \end{cases}$$

The right side contains the instructions for finding $v(t)$. The input t is converted into the output $+V$ or $-V$. The velocity $v(t)$ depends on t. In this case the function is "discontinuous," because the needle jumps at $t = 3$. *The velocity is not defined at that instant.* There is no $v(3)$. (You might argue that v is zero at the jump, but that leads to trouble.) The graph of f has a corner, and we can't give its slope.

The problem also involves a second function, namely the distance. The principle behind $f(t)$ is the same: $f(t)$ *is the distance at time* t. It is the net distance forward, and again the instructions change at $t = 3$. In the forward motion, $f(t)$ equals Vt as before. In the backward half, a calculation is built into the formula for $f(t)$:

$$f(t) = \begin{cases} Vt & \text{if} \quad 0 \leqslant t \leqslant 3 \\ V(6 - t) & \text{if} \quad 3 \leqslant t \leqslant 6 \end{cases}$$

At the switching time the right side gives two instructions (one on each line). This would be bad except that they agree: $f(3) = 3V$.† The distance function is "*con-*

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†A function is only allowed *one value* $f(t)$ or $v(t)$ at each time t.

tinuous." There is no jump in f, even when there is a jump in v. After $t = 3$ the distance decreases because of $-Vt$. At $t = 6$ the second instruction correctly gives $f(6) = 0$.

Notice something more. The functions were given by graphs before they were given by formulas. The graphs tell you f and v at every time t—sometimes more clearly than the formulas. The values $f(t)$ and $v(t)$ can also be given by tables or equations or a set of instructions. (In some way all functions are instructions—the function tells how to find f at time t.) Part of knowing f is knowing all its inputs and outputs—its *domain* and *range*:

The domain of a function is the set of inputs. The range is the set of outputs.

The domain of f consists of all times $0 \leqslant t \leqslant 6$. The range consists of all distances $0 \leqslant f(t) \leqslant 3V$. (The range of v contains only the two velocities $+V$ and $-V$.) We mention now, and repeat later, that every "linear" function has a formula $f(t) = vt + C$. Its graph is a line and v is the slope. The constant C moves the line up and down. It adjusts the line to go through any desired starting point.

SUMMARY: MORE ABOUT FUNCTIONS

May I collect together the ideas brought out by this example? We had two functions v and f. One was *velocity*, the other was *distance*. Each function had a *domain*, and a *range*, and most important a *graph*. For the f-graph we studied the slope (which agreed with v). For the v-graph we studied the area (which agreed with f). Calculus produces functions in pairs, and the best thing a book can do early is to show you more of them.

$$\begin{matrix} in \\ the \\ domain \end{matrix} \left\{ \begin{matrix} input\ t & \rightarrow & function\ f & \rightarrow & output\ f(t) \\ input\ 2 & \rightarrow & function\ v & \rightarrow & output\ v(2) \\ input\ 7 & \rightarrow & f(t) = 2t + 6 & \rightarrow & f(7) = 20 \end{matrix} \right\} \begin{matrix} in \\ the \\ range \end{matrix}$$

Note about the definition of a function. The idea behind the symbol $f(t)$ is absolutely crucial to mathematics. Words don't do it justice! By definition, a function is a "rule" that assigns one member of the range to each member of the domain. Or, a function is a set of pairs $(t, f(t))$ with no t appearing twice. (These are "ordered pairs" because we write t before $f(t)$.) Both of those definitions are correct—but somehow they are too passive.

In practice what matters is the active part. The number $f(t)$ is produced from the number t. We read a graph, plug into a formula, solve an equation, run a computer program. The input t is "mapped" to the output $f(t)$, which changes as t changes. Calculus is about the *rate of change*. This rate is our other function v.
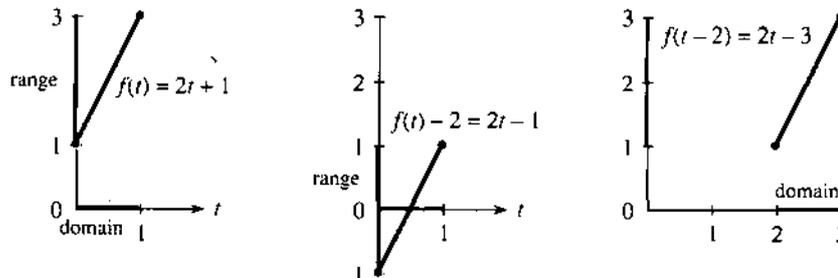
Fig. 1.5 Subtracting 2 from f affects the range. Subtracting 2 from t affects the domain.

It is quite hard at the beginning, and not automatic, to see the difference between $f(t) - 2$ and $f(t - 2)$. Those are both new functions, created out of the original $f(t)$. In $f(t) - 2$, we subtract 2 from all the distances. That moves the whole graph *down*. In $f(t - 2)$, we subtract 2 from the time. That moves the graph over *to the right*. Figure 1.5 shows both movements, starting from $f(t) = 2t + 1$. The formula to find $f(t - 2)$ is $2(t - 2) + 1$, which is $2t - 3$.

A graphing calculator also moves the graph, when you change the viewing window. You can pick any rectangle $A \leqslant t \leqslant B$, $C \leqslant f(t) \leqslant D$. The screen shows that part of the graph. But on the calculator, *the function $f(t)$ remains the same*. It is the axes that get renumbered. In our figures the axes stay the same and the function is changed.

There are two more basic ways to change a function. (We are always creating new functions—that is what mathematics is all about.) Instead of subtracting or adding, we can *multiply* the distance by 2. Figure 1.6 shows $2f(t)$. And instead of shifting the time, we can *speed it up*. The function becomes $f(2t)$. Everything happens twice as fast (and takes half as long). On the calculator those changes correspond to a "*zoom*"—on the f axis or the t axis. We soon come back to zooms.
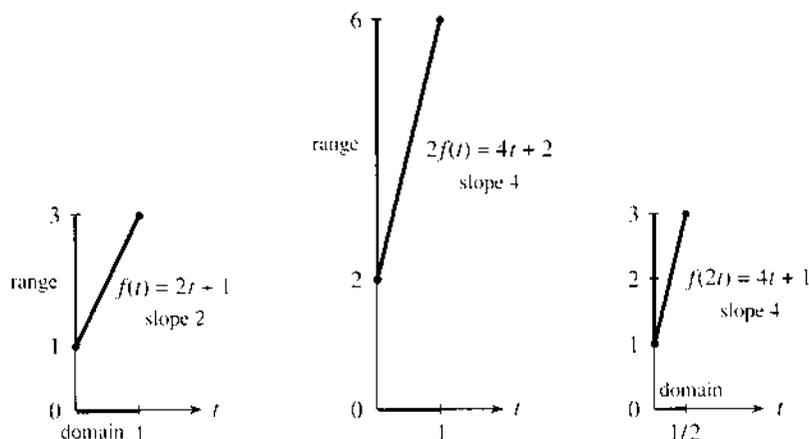
Fig. 1.6   Doubling the distance or speeding up the time doubles the slope.

1.1   EXERCISES

Each section of the book contains read-through questions. They allow you to outline the section yourself—more actively than reading a summary. This is probably the best way to remember the important ideas.

Starting from $f(0) = 0$ at constant velocity v, the distance function is $f(t) = $ __a__ . When $f(t) = 55t$ the velocity is $v = $ __b__ . When $f(t) = 55t + 1000$ the velocity is still __c__ and the starting value is $f(0) = $ __d__ . In each case v is the __e__ of the graph of f. When __f__ is negative, the graph of __g__ goes downward. In that case area in the v-graph counts as __h__ .

Forward motion from $f(0) = 0$ to $f(2) = 10$ has $v = $ __i__ . Then backward motion to $f(4) = 0$ has $v = $ __j__ . The distance function is $f(t) = 5t$ for $0 \leqslant t \leqslant 2$ and then $f(t) = $ __k__

(not $-5t$). The slopes are __l__ and __m__ . The distance $f(3) = $ __n__ . The area under the v-graph up to time 1.5 is __o__ . The domain of f is the time interval __p__ , and the range is the distance interval __q__ . The range of $v(t)$ is only __r__ .

The value of $f(t) = 3t + 1$ at $t = 2$ is $f(2) = $ __s__ . The value 19 equals $f($ __t__ $)$. The difference $f(4) - f(1) = $ __u__ . That is the change in distance, when $4 - 1$ is the change in __v__ . The ratio of those changes equals __w__ , which is the __x__ of the graph. The formula for $f(t) + 2$ is $3t + 3$ whereas $f(t + 2)$ equals __y__ . Those functions have the same __z__ as f: the graph of $f(t) + 2$ is shifted __A__ and $f(t + 2)$ is shifted __B__ . The formula for $f(5t)$ is __C__ . The formula for $5f(t)$ is __D__ . The slope has jumped from 3 to __E__ .

The set of inputs to a function is its __F__. The set of outputs is its __G__. The functions $f(t) = 7 + 3(t - 2)$ and $f(t) = vt + C$ are __H__. Their graphs are __I__ with slopes equal to __J__ and __K__. They are the same function, if $v = $ __L__ and $C = $ __M__.

Draw the velocity graph that goes with each distance graph.

1

1a

1b

2

2a

2b
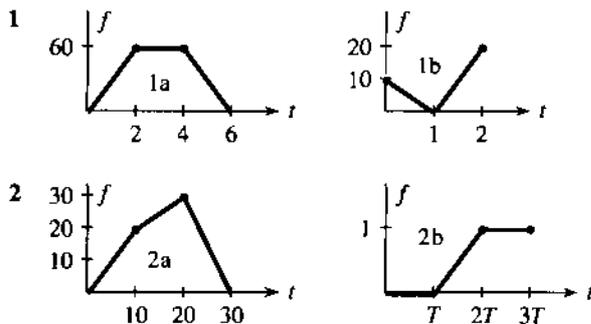
3 Write down three-part formulas for the velocities $v(t)$ in Problem 2, starting from $v(t) = 2$ for $0 < t < 10$.

4 The distance in 1b starts with $f(t) = 10 - 10t$ for $0 \leqslant t \leqslant 1$. Give a formula for the second part.

5 In the middle of graph 2a find $f(15)$ and $f(12)$ and $f(t)$.

6 In graph 2b find $f(1.4T)$. If $T = 3$ what is $f(4)$?

7 Find the *average speed* between $t = 0$ and $t = 5$ in graph 1a. What is the speed at $t = 5$?

8 What is the average speed between $t = 0$ and $t = 2$ in graph 1b? The average speed is zero between $t = \frac{1}{2}$ and $t = $ _____.

9 (recommended) A car goes at speed $v = 20$ into a brick wall at distance $f = 4$. Give two-part formulas for $v(t)$ and $f(t)$ (before and after), and draw the graphs.

10 Draw any reasonable graphs of $v(t)$ and $f(t)$ when
  (a) the driver backs up, stops to shift gear, then goes fast;
  (b) the driver slows to 55 for a police car;
  (c) in a rough gear change, the car accelerates in jumps;
  (d) the driver waits for a light that turns green.

11 Your bank account earns simple interest on the opening balance $f(0)$. What are the interest rates per year?
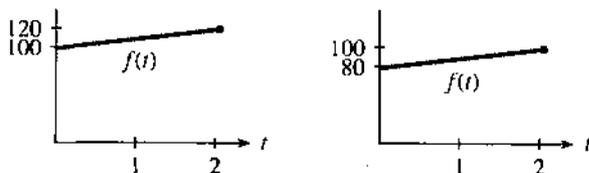
12 The earth's population is growing at $v = 100$ million a year, starting from $f = 5.2$ billion in 1990. Graph $f(t)$ and find $f(2000)$.

Draw the distance graph that goes with each velocity graph. Start from $f = 0$ at $t = 0$ and mark the distance.

13

13a

13b

14

14a

14b
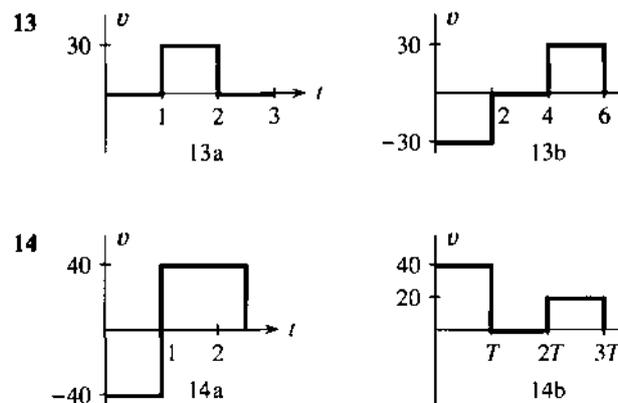
15 Write down formulas for $v(t)$ in Problem 14, starting with $v = -40$ for $0 < t < 1$. Find the average velocities to $t = 2.5$ and $t = 3T$.

16 Give 3-part formulas for the areas $f(t)$ under $v(t)$ in 13.

17 The distance in 14a starts with $f(t) = -40t$ for $0 \leqslant t \leqslant 1$. Find $f(t)$ in the other part, which passes through $f = 0$ at $t = 2$.

18 Draw the velocity and distance graphs if $v(t) = 8$ for $0 < t < 2$, $f(t) = 20 + t$ for $2 \leqslant t \leqslant 3$.

19 Draw rough graphs of $y = \sqrt{x}$ and $y = \sqrt{x - 4}$ and $y = \sqrt{x} - 4$. They are "half-parabolas" with infinite slope at the start.

20 What is the break-even point if x yearbooks cost $\$1200 + 30x$ to produce and the income is $40x$? The slope of the cost line is _____ (cost per additional book). If it goes above _____ you can't break even.

21 What are the domains and ranges of the distance functions in 14a and 14b—all values of t and $f(t)$ if $f(0) = 0$?

22 What is the range of $v(t)$ in 14b? Why is $t = 1$ not in the domain of $v(t)$ in 14a?

Problems 23–28 involve *linear functions* $f(t) = vt + C$. Find the constants v and C.

23 What linear function has $f(0) = 3$ and $f(2) = -11$?

24 Find *two* linear functions whose domain is $0 \leqslant t \leqslant 2$ and whose range is $1 \leqslant f(t) \leqslant 9$.

25 Find the linear function with $f(1) = 4$ and slope 6.

26 What functions have $f(t + 1) = f(t) + 2$?

27 Find the linear function with $f(t + 2) = f(t) + 6$ and $f(1) = 10$.

28 Find the only $f = vt$ that has $f(2t) = 4f(t)$. Show that every $f = \frac{1}{2}at^2$ has this property. To go _____ times as far in twice the time, you must accelerate.

29 Sketch the graph of $f(t) = |5 - 2t|$ (absolute value) for $|t| \leqslant 2$ and find its slopes and range.

30 Sketch the graph of $f(t) = 4 - t - |4 - t|$ for $2 \leqslant t \leqslant 5$ and find its slope and range.

31 Suppose $v = 8$ up to time T, and after that $v = -2$. Starting from zero, when does f return to zero? Give formulas for $v(t)$ and $f(t)$.

32 Suppose $v = 3$ up to time $T = 4$. What new velocity will lead to $f(7) = 30$ if $f(0) = 0$? Give formulas for $v(t)$ and $f(t)$.

33 What function F(C) converts Celsius temperature C to Fahrenheit temperature F? The slope is _____, which is the number of Fahrenheit degrees equivalent to $1°C$.

34 What function C(F) converts Fahrenheit to Celsius (or Centigrade), and what is its slope?

35 What function converts the weight w in grams to the weight $f(w)$ in kilograms? Interpret the slope of $f(w)$.

36 (Newspaper of March 1989) Ten hours after the accident the alcohol reading was .061. Blood alcohol is eliminated at .015 per hour. What was the reading at the time of the accident? How much later would it drop to .04 (the maximum set by the Coast Guard)? The usual limit on drivers is .10 percent.

Which points between $t = 0$ and $t = 5$ can be in the domain of $f(t)$? With this domain find the range in 37–42.

37 $f(t) = \sqrt{t - 1}$ **38** $f(t) = 1/\sqrt{t - 1}$

39 $f(t) = |t - 4|$ (absolute value)   **40** $f(t) = 1/(t - 4)^2$

41 $f(t) = 2^t$ **42** $f(t) = 2^{-t}$

43 (a) Draw the graph of $f(t) = \frac{1}{2}t + 3$ with domain $0 \leqslant t \leqslant 2$. Then give a formula and graph for

  (b) $f(t) + 1$ (c) $f(t + 1)$

  (d) $4f(t)$ (e) $f(4t)$.

44 (a) Draw the graph of $U(t) = $ step function $= \{0$ for $t < 0$, 1 for $t \geqslant 0\}$. Then draw

  (b) $U(t) + 2$ (c) $U(t + 2)$

  (d) $3U(t)$ (e) $U(3t)$.

45 (a) Draw the graph of $f(t) = t + 1$ for $-1 \leqslant t \leqslant 1$. Find the domain, range, slope, and formula for

  (b) $2f(t)$ (c) $f(t - 3)$ (d) $-f(t)$ (e) $f(-t)$.

46 If $f(t) = t - 1$ what are $2f(3t)$ and $f(1 - t)$ and $f(t - 1)$?

47 In the forward-back example find $f(\frac{1}{2}T)$ and $f(\frac{3}{2}T)$. Verify that those agree with the areas "under" the v-graph in Figure 1.4.

48 Find formulas for the outputs $f_1(t)$ and $f_2(t)$ which come from the input t:

  (1) inside $=$ input$*3$ (2) inside $\leftarrow$ input $+ 6$
     output $=$ inside $+ 3$ output $\leftarrow$ inside$*3$

Note BASIC and FORTRAN (and calculus itself) use $=$ instead of $\leftarrow$. But the symbol $\leftarrow$ or $:=$ is in some ways better. The instruction $t \leftarrow t + 6$ produces a new t equal to the old t plus six. The equation $t = t + 6$ is not intended.

49 Your computer can add and multiply. Starting with the number 1 and the input called t, give a list of instructions to lead to these outputs:

$$f_1(t) = t^2 + t \qquad f_2(t) = f_1(f_1(t)) \qquad f_3(t) = f_1(t + 1).$$

50 In fifty words or less explain what a *function* is.

The last questions are challenging but possible.

51 If $f(t) = 3t - 1$ for $0 \leqslant t \leqslant 2$ give formulas (with domain) and find the slopes of these six functions:

  (a) $f(t + 2)$ (b) $f(t) + 2$ (c) $2f(t)$

  (d) $f(2t)$ (e) $f(-t)$ (f) $f(f(t))$.

52 For $f(t) = vt + C$ find the formulas and slopes of

  (a) $3f(t) + 1$ (b) $f(3t + 1)$ (c) $2f(4t)$

  (d) $f(-t)$ (e) $f(t) - f(0)$ (f) $f(f(t))$.

53 (hardest) The forward-back function is $f(t) = 2t$ for $0 \leqslant t \leqslant 3$, $f(t) = 12 - 2t$ for $3 \leqslant t \leqslant 6$. Graph $f(f(t))$ and find its *four-part* formula. First try $t = 1.5$ and 3.

54 (a) Why is the letter **X** not the graph of a function?

  (b) Which capital letters are the graphs of functions?

  (c) Draw graphs of their slopes.

1.2 Calculus Without Limits

The next page is going to reveal one of the key ideas behind calculus. The discussion is just about numbers—functions and slopes can wait. The numbers are not even special, they can be any numbers. The crucial point is to look at their differences:

Suppose the numbers are $f = $ 0   2   6   7   4   9
Their differences are $v = $ 2   4   1 -3 5

The differences are printed in between, to show $2 - 0 = 2$ and $6 - 2 = 4$ and $7 - 6 = 1$.

Notice how $4 - 7$ gives a negative answer -3. The numbers in f can go up or down, the differences in v can be positive or negative. The idea behind calculus comes when you **add up those differences**:

$$2 + 4 + 1 - 3 + 5 = 9$$

The sum of differences is 9. This is the last number on the top line (in f). Is this an accident, or is this always true? If we stop earlier, after $2 + 4 + 1$, we get the 7 in f. Test any prediction on a second example:

$$\text{Suppose the numbers are } f = 1 \quad 3 \quad 7 \quad 8 \quad 5 \quad 10$$
$$\text{Their differences are } v = \qquad 2 \quad 4 \quad 1 \quad -3 \quad 5$$

The f's are increased by 1. **The differences are exactly the same**—no change. The sum of differences is still 9. But the last f is now 10. That prediction is not right, we don't always get the last f.

The first f is now 1. The answer 9 (the sum of differences) is $10 - 1$, **the last f minus the first f.** What happens when we change the f's in the middle?

$$\text{Suppose the numbers are } f = 1 \quad 5 \quad 12 \quad 7 \quad 10$$
$$\text{Their differences are } v = \qquad 4 \quad 7 \quad -5 \quad 3$$

The differences add to $4 + 7 - 5 + 3 = 9$. This is still $10 - 1$. No matter what f's we choose or how many, the sum of differences is controlled by the first f and last f. If this is always true, there must be a clear reason why *the middle f's cancel out*.

The sum of differences is $(5 - 1) + (12 - 5) + (7 - 12) + (10 - 7) = 10 - 1.$

The 5's cancel, the 12's cancel, and the 7's cancel. It is only $10 - 1$ that doesn't cancel. This is the key to calculus!

1B   The differences of the f's add up to $(f_{\text{last}} - f_{\text{first}})$.

EXAMPLE 1   The numbers grow linearly: $f = 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7$
Their differences are constant: $v = 1 \quad 1 \quad 1 \quad 1 \quad 1$

The sum of differences is certainly 5. This agrees with $7 - 2 = f_{\text{last}} - f_{\text{first}}$. The numbers in v remind us of constant velocity. The numbers in f remind us of a straight line $f = vt + C$. This example has $v = 1$ and the f's start at 2. The straight line would come from $f = t + 2$.

EXAMPLE 2   The numbers are squares: $f = \quad 0 \quad 1 \quad 4 \quad 9 \quad 16$
Their differences grow linearly: $v = 1 \quad 3 \quad 5 \quad 7$

$1 + 3 + 5 + 7$ agrees with $4^2 = 16$. It is a beautiful fact that the first j odd numbers always add up to j^2. The v's are the odd numbers, the f's are perfect squares.

Note   The letter j is sometimes useful to tell which number in f we are looking at. For this example the zeroth number is $f_0 = 0$ and the jth number is $f_j = j^2$. This is a part of algebra, to give a formula for the f's instead of a list of numbers. We can also use j to tell which difference we are looking at. The first v is the first odd number $v_1 = 1$. The jth difference is the jth odd number $v_j = 2j - 1$. (Thus v_4 is $8 - 1 = 7$.) It is better to start the differences with $j = 1$, since there is no zeroth odd number v_0.

With this notation the jth **difference is** $v_j = f_j - f_{j-1}$. Sooner or later you will get comfortable with subscripts like j and $j - 1$, but it can be later. The important point is that the sum of the v's equals $f_{\text{last}} - f_{\text{first}}$. We now connect the v's to slopes and the f's to areas.
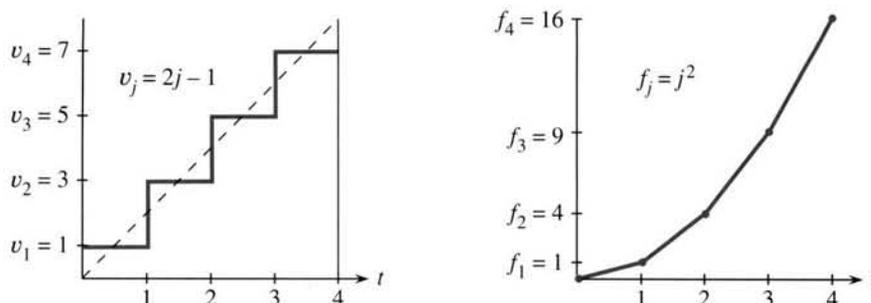
Fig. 1.7 Linear increase in $v = 1, 3, 5, 7$. Squares in the distances $f = 0, 1, 4, 9, 16$.

Figure 1.7 shows a natural way to graph Example 2, with the odd numbers in v and the squares in f. Notice an important difference between the v-graph and the f-graph. The graph of f is "*piecewise linear*." We plotted the numbers in f and connected them by straight lines. The graph of v is "*piecewise constant*." We plotted the differences as constant over each piece. This reminds us of the distance-velocity graphs, when the distance $f(t)$ is a straight line and the velocity $v(t)$ is a horizontal line.

Now make the connection to slopes:

$$\text{The slope of the } f\text{-graph is } \frac{\text{distance up}}{\text{distance across}} = \frac{\text{change in } f}{\text{change in } t} = v.$$

Over each piece, the change in t (across) is 1. The change in f (upward) is the difference that we are calling v. The ratio is the slope $v/1$ or just v. The slope makes a sudden change at the breakpoints $t = 1, 2, 3, \ldots$. At those special points the slope of the f-graph is not defined—we connected the v's by vertical lines but this is very debatable. *The main idea is that between the breakpoints, the slope of $f(t)$ is $v(t)$*.

Now make the connection to areas:

The total area under the v-graph is $f_{\text{last}} - f_{\text{first}}$.

This area, underneath the staircase in Figure 1.7, is composed of rectangles. The base of every rectangle is 1. The heights of the rectangles are the v's. So the areas also equal the v's, and the total area is the sum of the v's. This area is $f_{\text{last}} - f_{\text{first}}$.

Even more is true. We could start at any time and end at any later time —not necessarily at the special times $t = 0, 1, 2, 3, 4$. Suppose we stop at $t = 3.5$. Only half of the last rectangular area (under $v = 7$) will be counted. The total area is $1 + 3 + 5 + \frac{1}{2}(7) = 12.5$. This still agrees with $f_{\text{last}} - f_{\text{first}} = 12.5 - 0$. At this new ending time $t = 3.5$, we are only halfway up the last step in the f-graph. Halfway between 9 and 16 is 12.5.

> **1C** *The v's are slopes of $f(t)$. The area under the v-graph is $f(t_{\text{end}}) - f(t_{\text{start}})$.*

This is nothing less than the Fundamental Theorem of Calculus. But we have only used algebra (no curved graphs and no calculations involving limits). For now the Theorem is restricted to piecewise linear $f(t)$ and piecewise constant $v(t)$. In Chapter 5 that restriction will be overcome.

Notice that a proof of $1 + 3 + 5 + 7 = 4^2$ is suggested by Figure 1.7a. The triangle under the dotted line has the same area as the four rectangles under the staircase. The area of the triangle is $\frac{1}{2} \cdot \text{base} \cdot \text{height} = \frac{1}{2} \cdot 4 \cdot 8$, which is the perfect square 4^2. When there are j rectangles instead of 4, we get $\frac{1}{2} \cdot j \cdot 2j = j^2$ for the area.

The next examples show other patterns, where f and v increase exponentially or oscillate around zero. I hope you like them but I don't think you have to learn them. They are like the special functions 2^t and $\sin t$ and $\cos t$—except they go in steps. You get a first look at the important functions of calculus, but you only need algebra. *Calculus is needed for a steadily changing velocity, when the graph of f is curved.*

The last example will be *income tax*—which really does go. in steps. Then Section 1.3 will introduce the slope of a curve. The crucial step for curves is working with *limits*. That will take us from algebra to calculus.

EXPONENTIAL VELOCITY AND DISTANCE

Start with the numbers $f = 1, 2, 4, 8, 16$. These are "powers of 2." They start with the zeroth power, which is $2^0 = 1$. *The exponential starts at 1 and not 0.* After j steps there are j factors of 2, and f_j equals 2^j. *Please recognize the difference between $2j$ and j^2 and 2^j.* The numbers $2j$ grow linearly, the numbers j^2 grow quadratically, the numbers 2^j grow exponentially. At $j = 10$ these are 20 and 100 and 1024. The exponential 2^j quickly becomes much larger than the others.

The differences of $f = 1, 2, 4, 8, 16$ are exactly $v = 1, 2, 4, 8$. We get the same beautiful numbers. *When the f's are powers of 2, so are the v's.* The formula $v_j = 2^{j-1}$ is slightly different from $f_j = 2^j$, because the first v is numbered v_1. (Then $v_1 = 2^0 = 1$. The zeroth power of every number is 1, except that 0^0 is meaningless.) The two graphs in Figure 1.8 use the same numbers but they look different, because f is piecewise linear and v is piecewise constant.
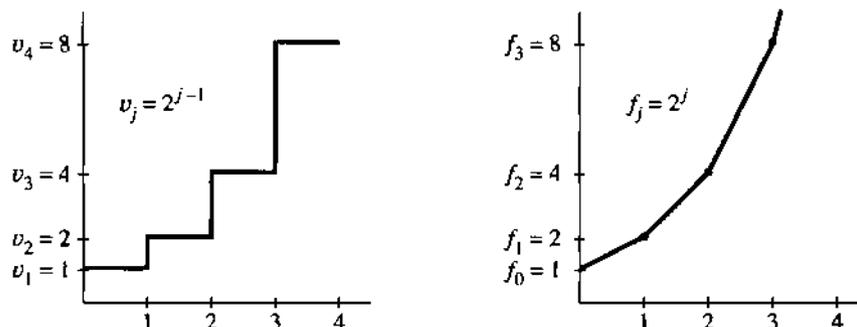
Fig. 1.8 The velocity and distance grow exponentially (powers of 2).

Where will calculus come in? It works with the smooth curve $f(t) = 2^t$. This exponential growth is critically important for population and money in a bank and the national debt. You can spot it by the following test: $v(t)$ **is proportional to** $f(t)$.

Remark The function 2^t is trickier than t^2. For $f = t^2$ the slope is $v = 2t$. It is proportional to t and not t^2. For $f = 2^t$ the slope is $v = c2^t$, and we won't find the constant $c = .693 \ldots$ until Chapter 6. (The number c is the natural logarithm of 2.) Problem 37 estimates c with a calculator—the important thing is that it's constant.

OSCILLATING VELOCITY AND DISTANCE

We have seen a forward-back motion, velocity V followed by $-V$. That is oscillation of the simplest kind. The graph of f goes linearly up and linearly down. Figure 1.9 shows another oscillation that returns to zero, but the path is more interesting.

The numbers in f are now $0, 1, 1, 0, -1, -1, 0$. Since $f_6 = 0$ the motion brings us back to the start. The whole oscillation can be repeated.

The differences in v are $1, 0, -1, -1, 0, 1$. They add up to zero, which agrees with $f_{last} - f_{first}$. It is the same oscillation as in f (and also repeatable), but shifted in time.

The f-graph resembles (roughly) a *sine curve*. The v-graph resembles (even more roughly) a *cosine curve*. The waveforms in nature are smooth curves, while these are "digitized"—the way a digital watch goes forward in jumps. You recognize that the change from analog to digital brought the computer revolution. The same revolution is coming in CD players. Digital signals (off or on, 0 or 1) seem to win every time.

The piecewise v and f start again at $t = 6$. The ordinary sine and cosine repeat at $t = 2\pi$. A repeating motion is *periodic*—here the "period" is 6 or 2π. (With t in degrees the period is 360—a full circle. The period becomes 2π when angles are measured in *radians*. We virtually always use radians—which are degrees times $2\pi/360$.) A watch has a period of 12 hours. If the dial shows AM and PM, the period is _____.
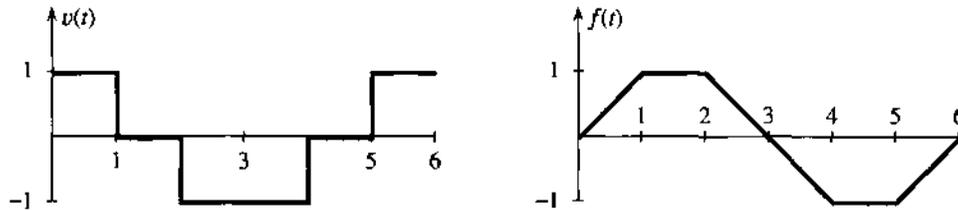
Fig. 1.9 Piecewise constant "cosine" and piecewise linear "sine." They both repeat.

A SHORT BURST OF SPEED

The next example is a car that is driven fast for a short time. The speed is V until the distance reaches $f = 1$, when the car suddenly stops. The graph of f goes up linearly with slope V, and then across with slope zero:

$$v(t) = \begin{cases} V & \text{up to} \quad t = T \\ 0 & \text{after} \quad t = T \end{cases} \qquad f(t) = \begin{cases} Vt & \text{up to} \quad t = T \\ 1 & \text{after} \quad t = T \end{cases}$$

This is another example of "function notation." Notice the general time t and the particular stopping time T. The distance is $f(t)$. The domain of f (the inputs) includes all times $t \geqslant 0$. The range of f (the outputs) includes all distances $0 \leqslant f \leqslant 1$.

Figure 1.10 allows us to compare three cars—a Jeep and a Corvette and a Maserati. They have different speeds but they all reach $f = 1$. So the areas under the v-graphs are all 1. The rectangles have height V and base $T = 1/V$.
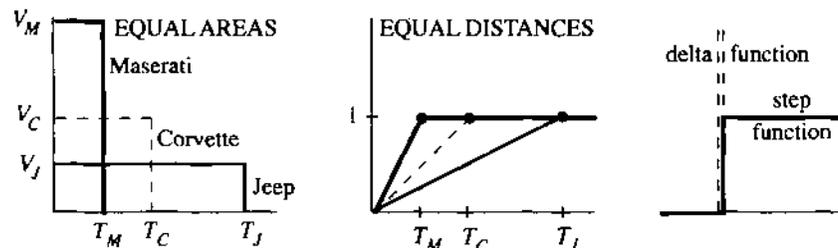
Fig. 1.10 Bursts of speed with $V_M T_M = V_C T_C = V_J T_J = 1$. Step function has infinite slope.

Optional remark It is natural to think about faster and faster speeds, which means steeper slopes. The f-graph reaches 1 in shorter times. The extreme case is a *step function*, when the graph of f goes straight up. This is the unit step $U(t)$, which is zero up to $t = 0$ and jumps immediately to $U = 1$ for $t > 0$.

What is the slope of the step function? It is zero except at the jump. At that moment, which is $t = 0$, the slope is *infinite*. We don't have an ordinary velocity $v(t)$—instead we have an impulse that makes the car jump. The graph is a spike over the single point $t = 0$, and it is often denoted by δ—so the slope of the step function is called a "*delta function.*" The area under the infinite spike is 1.

You are absolutely not responsible for the theory of delta functions! Calculus is about curves, not jumps.

Our last example is a real-world application of slopes and rates—to explain "how taxes work." Note especially the difference between tax rates and tax brackets and total tax. The rates are v, the brackets are on x, the total tax is f.

EXAMPLE 3 *Income tax is piecewise linear. The slopes are the tax rates* .15, .28, .31.

Suppose you are single with taxable income of x dollars (Form 1040, line 37—after all deductions). These are the 1991 instructions from the Internal Revenue Service:

If x is not over \$20,350, the tax is 15% of x.

If \$20,350 $\leqslant x \leqslant$ \$49,300, the tax is \$3052.50 + 28% of the amount over \$20,350.

If x is over \$49,300, the tax is \$11,158.50 + 31% of the amount over \$49,300.

The first bracket is $0 \leqslant x \leqslant$ \$20,350. (The IRS never uses this symbol $\leqslant$, but I think it is OK here. We know what it means.) The second bracket is \$20,350 $\leqslant x \leqslant$ \$49,300. The top bracket $x \geqslant$ \$49,300 pays tax at the top rate of 31%. But only the income *in that bracket* is taxed at that rate.

Figure 1.11 shows the rates and the brackets and the tax due. Those are not *average* rates, they are *marginal* rates. Total tax divided by total income would be the average rate. The marginal rate of .28 or .31 gives the tax on each *additional* dollar of income— it is the slope at the point x. *Tax is like* area *or* distance—it adds up. *Tax rate is like* slope *or* velocity—it depends where you are. This is often unclear in the news media.
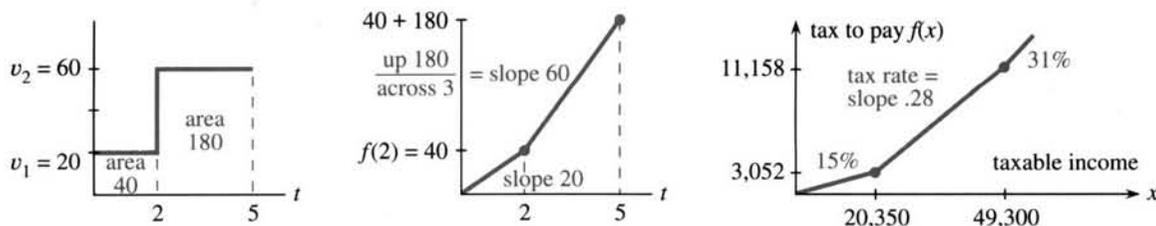
Fig. 1.11 The tax rate is v, the total tax is f. Tax brackets end at breakpoints.

Question What is the equation for the straight line in the top bracket?
Answer The bracket begins at $x =$ \$49,300 when the tax is $f(x) =$ \$11,158.50. The slope of the line is the tax rate .31. When we know a point on the line and the slope, we know the equation. This is important enough to be highlighted.

> **1D** For x in the top bracket the tax is $f(x) =$ \$11,158.50 + .31 $(x -$ \$49,300). This is the tax on \$49,300 plus the extra tax on extra income.

Section 2.3 presents this "point-slope equation" for any straight line. Here you see it for one specific example. Where does the number \$11,158.50 come from? It is the tax at the *end* of the middle bracket, so it is the tax at the *start* of the top bracket.

Figure 1.11 also shows a distance-velocity example. The distance at $t = 2$ is $f(2) = 40$ miles. After that time the velocity is 60 miles per hour. So the line with slope 60 on the f-graph has the equation

$$f(t) = \text{starting distance} + \text{extra distance} = 40 + 60(t - 2).$$

The starting point is $(2, 40)$. The new speed 60 multiplies the extra time $t - 2$. The point-slope equation makes sense. *We now review this section, with comments.*

Central idea   Start with any numbers in f. Their differences go in v. Then the sum of those differences is $f_{last} - f_{first}$.

Subscript notation   The numbers are $f_0, f_1, \ldots$ and the first difference is $v_1 = f_1 - f_0$. A typical number is f_j and the jth difference is $v_j = f_j - f_{j-1}$. When those differences are added, all f's in the middle (like f_1) cancel out:

$$v_1 + v_2 + \cdots + v_j = (f_1 - f_0) + (f_2 - f_1) + \cdots + (f_j - f_{j-1}) = f_j - f_0.$$

Examples $f_j = j$ or j^2 or 2^j. Then $v_j = 1$ (constant) or $2j - 1$ (odd numbers) or 2^{j-1}.

Functions   Connect the f's to be piecewise linear. Then the slope v is piecewise constant. The area under the v-graph from any t_{start} to any t_{end} equals $f(t_{end}) - f(t_{start})$.

Units   Distance in miles and velocity in miles per hour. Tax in dollars and tax rate in (dollars paid)/(dollars earned). Tax rate is a percentage like .28, with no units.

1.2   EXERCISES

Read-through questions

Start with the numbers $f = 1, 6, 2, 5$. Their differences are $v =$ __a__ . The sum of those differences is __b__ . This is equal to f_{last} minus __c__ . The numbers 6 and 2 have no effect on this answer, because in $(6 - 1) + (2 - 6) + (5 - 2)$ the numbers 6 and 2 __d__ . The slope of the line between $f(0) = 1$ and $f(1) = 6$ is __e__ . The equation of that line is $f(t) =$ __f__ .

With distances 1, 5, 25 at unit times, the velocities are __g__ . These are the __h__ of the f-graph. The slope of the tax graph is the tax __i__ . If $f(t)$ is the postage cost for t ounces or t grams, the slope is the __j__ per __k__ . For distances 0, 1, 4, 9 the velocities are __l__ . The sum of the first j odd numbers is $f_j =$ __m__ . Then f_{10} is __n__ and the velocity v_{10} is __o__ .

The piecewise linear sine has slopes __p__ . Those form a piecewise __q__ cosine. Both functions have __r__ equal to 6, which means that $f(t + 6) =$ __s__ for every t. The velocities $v = 1, 2, 4, 8, \ldots$ have $v_j =$ __t__ . In that case $f_0 = 1$ and $f_j =$ __u__ . The sum of $1, 2, 4, 8, 16$ is __v__ . The difference $2^j - 2^{j-1}$ equals __w__ . After a burst of speed V to time T, the distance is __x__ . If $f(T) = 1$ and V increases, the burst lasts only to $T =$ __y__ . When V approaches infinity, $f(t)$ approaches a __z__ function. The velocities approach a __A__ function, which is concentrated at $t = 0$ but has area __B__ under its graph. The slope of a step function is __C__ .

Problems 1–4 are about numbers f and differences v.

1 From the numbers $f = 0, 2, 7, 10$ find the differences v and the sum of the three v's. Write down another f that leads to the same v's. For $f = 0, 3, 12, 10$ the sum of the v's is still _____.

2 Starting from $f = 1, 3, 2, 4$ draw the f-graph (linear pieces) and the v-graph. What are the areas "under" the v-graph that add to $4 - 1$? If the next number in f is 11, what is the area under the next v?

3 From $v = 1, 2, 1, 0, -1$ find the f's starting at $f_0 = 3$. Graph v and f. The maximum value of f occurs when $v =$ _____. Where is the maximum f when $v = 1, 2, 1, -1$?

4 For $f = 1, b, c, 7$ find the differences v_1, v_2, v_3 and add them up. Do the same for $f = a, b, c, 7$. Do the same for $f = a, b, c, d$.

Problems 5–11 are about linear functions and constant slopes.

5 Write down the slopes of these linear functions:

  (a) $f(t) = 1.1t$ (b) $f(t) = 1 - 2t$ (c) $f(t) = 4 + 5(t - 6)$.

Compute $f(6)$ and $f(7)$ for each function and confirm that $f(7) - f(6)$ equals the slope.

6 If $f(t) = 5 + 3(t - 1)$ and $g(t) = 1.5 + 2.5(t - 1)$ what is $h(t) = f(t) - g(t)$? Find the slopes of f, g, and h.

7 Suppose $v(t) = 2$ for $t < 5$ and $v(t) = 3$ for $t > 5$.

(a) If $f(0) = 0$ find a two-part formula for $f(t)$.

(b) Check that $f(10)$ equals the area under the graph of $v(t)$ (two rectangles) up to $t = 10$.

8 Suppose $v(t) = 10$ for $t < 1/10$, $v(t) = 0$ for $t > 1/10$. Starting from $f(0) = 1$ find $f(t)$ in two pieces.

9 Suppose $g(t) = 2t + 1$ and $f(t) = 4t$. Find $g(3)$ and $f(g(3))$ and $f(g(t))$. How is the slope of $f(g(t))$ related to the slopes of f and g?

10 For the same functions, what are $f(3)$ and $g(f(3))$ and $g(f(t))$? When t is changed to $4t$, distance increases _____ times as fast and the velocity is multiplied by _____.

11 Compute $f(6)$ and $f(8)$ for the functions in Problem 5. Confirm that the slopes v agree with

$$\text{slope} = \frac{f(8) - f(6)}{8 - 6} = \frac{\text{change in } f}{\text{change in } t}.$$

Problems 12–18 are based on Example 3 about income taxes.

12 What are the income taxes on $x = \$10,000$ and $x = \$30,000$ and $x = \$50,000$?

13 What is the equation for income tax $f(x)$ in the second bracket $\$20,350 \leqslant x \leqslant \$49,300$? How is the number 11,158.50 connected with the other numbers in the tax instructions?

14 Write the tax function $F(x)$ for a married couple if the IRS treats them as two single taxpayers each with taxable income $x/2$. (This is not done.)

15 In the 15% bracket, with 5% state tax as a deduction, the combined rate is not 20% but _____. Think about the tax on an extra \$100.

16 A piecewise linear function is *continuous* when $f(t)$ at the end of each interval equals $f(t)$ at the start of the following interval. If $f(t) = 5t$ up to $t = 1$ and $v(t) = 2$ for $t > 1$, define f beyond $t = 1$ so it is (a) continuous (b) discontinuous. (c) Define a tax function $f(x)$ with rates .15 and .28 so you would lose by earning an extra dollar beyond the breakpoint.

17 The difference between a tax *credit* and a *deduction* from income is the difference between $f(x) - c$ and $f(x - d)$. Which is more desirable, a credit of $c = \$1000$ or a deduction of $d = \$1000$, and why? Sketch the tax graphs when $f(x) = .15x$.

18 The average tax rate on the taxable income x is $a(x) = f(x)/x$. This is the slope between $(0, 0)$ and the point $(x, f(x))$. Draw a rough graph of $a(x)$. The average rate a is below the marginal rate v because _____.

Problems 19–30 involve numbers $f_0, f_1, f_2, \ldots$ and their differences $v_j = f_j - f_{j-1}$. They give practice with subscripts $0, \ldots, j$.

19 Find the velocities v_1, v_2, v_3 and formulas for v_j and f_j:

(a) $f = 1, 3, 5, 7 \ldots$ (b) $f = 0, 1, 0, 1, \ldots$ (c) $f = 0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots$

20 Find f_1, f_2, f_3 and a formula for f_j with $f_0 = 0$:

(a) $v = 1, 2, 4, 8, \ldots$ (b) $v = -1, 1, -1, 1, \ldots$

21 The areas of these nested squares are $1^2, 2^2, 3^2, \ldots$. What are the areas of the L-shaped bands (the differences between squares)? How does the figure show that $1 + 3 + 5 + 7 = 4^2$?
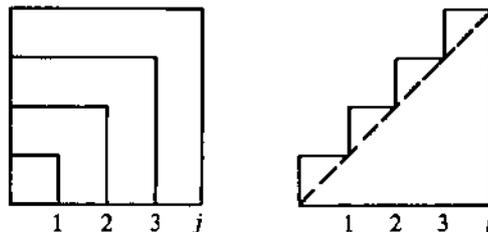
22 From the area under the staircase (by rectangles and then by triangles) show that the first j whole numbers 1 to j add up to $\frac{1}{2}j^2 + \frac{1}{2}j$. Find $1 + 2 + \cdots + 100$.

23 If $v = 1, 3, 5, \ldots$ then $f_j = j^2$. If $v = 1, 1, 1, \ldots$ then $f_j =$ _____. Add those to find the sum of $2, 4, 6, \ldots, 2j$. Divide by 2 to find the sum of $1, 2, 3, \ldots, j$. (Compare Problem 22.)

24 *True* (with reason) *or false* (with example).

(a) When the f's are increasing so are the v's.

(b) When the v's are increasing so are the f's.

(c) When the f's are periodic so are the v's.

(d) When the v's are periodic so are the f's.

25 If $f(t) = t^2$, compute $f(99)$ and $f(101)$. Between those times, what is the increase in f divided by the increase in t?

26 If $f(t) = t^2 + t$, compute $f(99)$ and $f(101)$. Between those times, what is the increase in f divided by the increase in t?

27 If $f_j = j^2 + j + 1$ find a formula for v_j.

28 Suppose the v's increase by 4 at every step. Show by example and then by algebra that the "second difference" $f_{j+1} - 2f_j + f_{j-1}$ equals 4.

29 Suppose $f_0 = 0$ and the v's are $1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \ldots$. For which j does $f_j = 5$?

30 Show that $a_j = f_{j+1} - 2f_j + f_{j-1}$ always equals $v_{j+1} - v_j$. If v is velocity then a stands for _____.

Problems 31–34 involve periodic f's and v's (like $\sin t$ and $\cos t$).

31 For the discrete sine $f = 0, 1, 1, 0, -1, -1, 0$ find the second differences $a_1 = f_2 - 2f_1 + f_0$ and $a_2 = f_3 - 2f_2 + f_1$ and a_3. Compare a_j with f_j.

32 If the sequence $v_1, v_2, \ldots$ has period 6 and $w_1, w_2, \ldots$ has period 10, what is the period of $v_1 + w_1, v_2 + w_2, \ldots$?

33 Draw the graph of $f(t)$ starting from $f_0 = 0$ when $v = 1, -1, -1, 1$. If v has period 4 find $f(12), f(13), f(100.1)$.

34 Graph $f(t)$ from $f_0 = 0$ to $f_4 = 4$ when $v = 1, 2, 1, 0$. If v has period 4, find $f(12)$ and $f(14)$ and $f(16)$. *Why doesn't f have period 4?*

Problems 35–42 are about exponential v's and f's.

35 Find the v's for $f = 1, 3, 9, 27$. Predict v_4 and v_j. Algebra gives $3^j - 3^{j-1} = (3-1)3^{j-1}$.

36 Find $1 + 2 + 4 + \cdots + 32$ and also $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{32}$.

37 Estimate the slope of $f(t) = 2^t$ at $t = 0$. Use a calculator to compute (increase in f)/(increase in t) when t is small:

$$\frac{f(t) - f(0)}{t} = \frac{2-1}{1} \text{ and } \frac{2^{.1}-1}{.1} \text{ and } \frac{2^{.01}-1}{.01} \text{ and } \frac{2^{.001}-1}{.001}.$$

38 Suppose $f_0 = 1$ and $v_j = 2f_{j-1}$. Find f_4.

39 (a) From $f = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ find v_1, v_2, v_3 and predict v_j.
   (b) Check $f_3 - f_0 = v_1 + v_2 + v_3$ and $f_j - f_{j-1} = v_j$.

40 Suppose $v_j = r^j$. Show that $f_j = (r^{j+1} - 1)/(r-1)$ starts from $f_0 = 1$ and has $f_j - f_{j-1} = v_j$. (Then this is the correct $f_j = 1 + r + \cdots + r^j = $ sum of a geometric series.)

41 From $f_j = (-1)^j$ compute v_j. What is $v_1 + v_2 + \cdots + v_j$?

42 Estimate the slope of $f(t) = e^t$ at $t = 0$. Use a calculator that knows e (or else take $e = 2.78$) to compute

$$\frac{f(t) - f(0)}{t} = \frac{e-1}{1} \text{ and } \frac{e^{.1}-1}{.1} \text{ and } \frac{e^{.01}-1}{.01}.$$

Problems 43–47 are about $U(t) = $ step from 0 to 1 at $t = 0$.

43 Graph the four functions $U(t-1)$ and $U(t)-2$ and $U(3t)$ and $4U(t)$. Then graph $f(t) = 4U(3t-1) - 2$.

44 Graph the square wave $U(t) - U(t-1)$. If this is the velocity $v(t)$, graph the distance $f(t)$. If this is the distance $f(t)$, graph the velocity.

45 Two bursts of speed lead to the same distance $f = 10$:

$v = $ _____ to $t = .001$　　$v = V$ to $t = $ _____.

As $V \to \infty$ the limit of the $f(t)$'s is _____.

46 Draw the staircase function $U(t) + U(t-1) + U(t-2)$. Its slope is a sum of three _____ functions.

47 Which capital letters like L are the graphs of functions when steps are allowed? The slope of L is minus a delta function. Graph the slopes of the others.

48 Write a subroutine FINDV whose input is a sequence $f_0, f_1, \ldots, f_N$ and whose output is $v_1, v_2, \ldots, v_N$. Include graphical output if possible. Test on $f_j = 2j$ and j^2 and 2^j.

49 Write a subroutine FINDF whose input is $v_1, \ldots, v_N$ and f_0, and whose output is $f_0, f_1, \ldots, f_N$. The default value of f_0 is zero. Include graphical output if possible. Test $v_j = j$.

50 If FINDV is applied to the output of FINDF, what sequence is returned? If FINDF is applied to the output of FINDV, what sequence is returned? Watch f_0.

51 Arrange $2j$ and j^2 and 2^j and $\sqrt{j}$ in increasing order
   (a) when j is large: $j = 9$　 (b) when j is small: $j = \frac{1}{9}$.

52 The average age of your family since 1970 is a piecewise linear function $A(t)$. Is it continuous or does it jump? What is its slope? Graph it the best you can.

1.3 The Velocity at an Instant

We have arrived at the central problems that calculus was invented to solve. There are two questions, in opposite directions, and I hope you could see them coming.

　1. If the velocity is changing, *how can you compute the distance traveled?*
　2. If the graph of $f(t)$ is not a straight line, *what is its slope?*

Find the distance from the velocity, find the velocity from the distance. Our goal is to do both—but not in one section. Calculus may be a good course, but it is not magic. The first step is to let the velocity change in the steadiest possible way.

Question 1 *Suppose the velocity at each time t is $v(t) = 2t$. Find $f(t)$.*

With $v = 2t$, a physicist would say that the acceleration is constant (it equals 2). The driver steps on the gas, the car accelerates, and the speedometer goes steadily up. The distance goes up too—faster and faster. If we measure t in seconds and v in feet per second, the distance f comes out in feet. After 10 seconds the speed is 20 feet per second. After 44 seconds the speed is 88 feet/second (which is 60 miles/hour). The acceleration is clear, *but how far has the car gone?*

Question 2 *The distance traveled by time t is* $f(t) = t^2$. Find the velocity $v(t)$.

The graph of $f(t) = t^2$ is on the right of Figure 1.12. It is a *parabola*. The curve starts at zero, when the car is new. At $t = 5$ the distance is $f = 25$. By $t = 10$, f reaches 100.

Velocity is distance divided by time, but what happens when the speed is changing? Dividing $f = 100$ by $t = 10$ gives $v = 10$—the *average velocity* over the first ten seconds. Dividing $f = 121$ by $t = 11$ gives the average speed over 11 seconds. But how do we find the *instantaneous velocity*—the reading on the speedometer at the exact instant when $t = 10$?
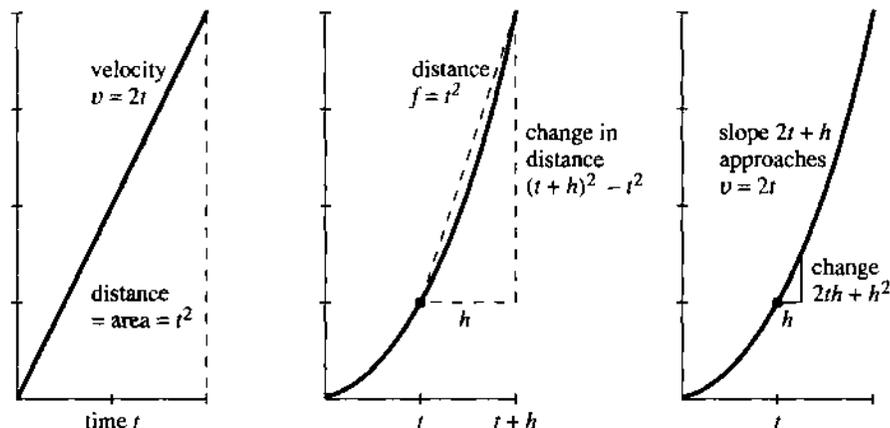
Fig. 1.12 The velocity $v = 2t$ is linear. The distance $f = t^2$ is quadratic.

I hope you see the problem. As the car goes faster, the graph of t^2 gets steeper—because more distance is covered in each second. The average velocity between $t = 10$ and $t = 11$ is a good approximation—but only an approximation—to the speed at the moment $t = 10$. Averages are easy to find:

distance at $t = 10$ is $f(10) = 10^2 = 100$ distance at $t = 11$ is $f(11) = 11^2 = 121$

$$\text{average velocity is } \frac{f(11) - f(10)}{11 - 10} = \frac{121 - 100}{1} = 21.$$

The car covered 21 feet in that 1 second. Its average speed was 21 feet/second. Since it was gaining speed, the velocity at the beginning of that second was below 21.

Geometrically, what is the average? It is a slope, but not the slope of the curve. *The average velocity is the slope of a straight line.* The line goes between two points on the curve in Figure 1.12. When we compute an average, we pretend the velocity is constant—so we go back to the easiest case. It only requires a division of distance by time:

$$\text{average velocity} = \frac{\text{change in } f}{\text{change in } t}. \tag{1}$$

Calculus and the Law You enter a highway at $1:00$. If you exit 150 miles away at $3:00$, your average speed is 75 miles per hour. I'm not sure if the police can give you a ticket. You could say to the judge, "When was I doing 75?" The police would have

to admit that they have no idea—but they would have a definite feeling that you must have been doing 75 sometime.†

We return to the central problem—computing $v(10)$ at the instant $t = 10$. The average velocity over the next second is 21. We can also find the average over the *half-second* between $t = 10.0$ and $t = 10.5$. Divide the change in distance by the change in time:

$$\frac{f(10.5) - f(10.0)}{10.5 - 10.0} = \frac{(10.5)^2 - (10.0)^2}{.5} = \frac{110.25 - 100}{.5} = 20.5.$$

That average of 20.5 is closer to the speed at $t = 10$. It is still not exact.

The way to find $v(10)$ is to *keep reducing the time interval*. This is the basis for Chapter 2, and the key to differential calculus. **Find the slope between points that are closer and closer on the curve.** The "limit" is the slope at a single point.

Algebra gives the average velocity between $t = 10$ and any later time $t = 10 + h$. The distance increases from 10^2 to $(10 + h)^2$. The change in time is h. So divide:

$$v_{\text{average}} = \frac{(10 + h)^2 - 10^2}{h} = \frac{100 + 20h + h^2 - 100}{h} = 20 + h. \qquad (2)$$

This formula fits our previous calculations. The interval from $t = 10$ to $t = 11$ had $h = 1$, and the average was $20 + h = 21$. When the time step was $h = \frac{1}{2}$, the average was $20 + \frac{1}{2} = 20.5$. Over a millionth of a second the average will be 20 plus $1/1,000,000$—which is very near 20.

Conclusion: *The velocity at $t = 10$ is $v = 20$. That is the slope of the curve.* It agrees with the v-graph on the left side of Figure 1.12, which also has $v(10) = 20$.

We now show that the two graphs match at all times. If $f(t) = t^2$ then $v(t) = 2t$. You are seeing the key computation of calculus, and we can put it into words before equations. Compute the distance at time $t + h$, *subtract* the distance at time t, and *divide* by h. That gives the average velocity:

$$v_{\text{ave}} = \frac{f(t + h) - f(t)}{h} = \frac{(t + h)^2 - t^2}{h} = \frac{t^2 + 2th + h^2 - t^2}{h} = 2t + h. \qquad (3)$$

This fits the previous calculation, where t was 10. The average was $20 + h$. Now the average is $2t + h$. It depends on the time step h, because the velocity is changing. But we can see what happens *as h approaches zero*. The average is closer and closer to the speedometer reading of $2t$, at the exact moment when the clock shows time t:

1E As h approaches zero, the average velocity $2t + h$ approaches $v(t) = 2t$.

Note The computation (3) shows how calculus needs algebra. If we want the whole v-graph, we have to let time be a "*variable*." It is represented by the letter t. Numbers are enough at the specific time $t = 10$ and the specific step $h = 1$—but algebra gets beyond that. The average between any t and any $t + h$ is $2t + h$. Please don't hesitate to put back numbers for the letters—that checks the algebra.

---

†This is our first encounter with the much despised "Mean Value Theorem." If the judge can prove the theorem, you are dead. A few v-graphs and f-graphs will confuse the situation (possibly also a delta function).

There is also a step beyond algebra! Calculus requires the *limit of the average*. As h shrinks to zero, the points on the graph come closer. "Average over an interval" becomes "velocity at an instant." The general theory of limits is not particularly simple, but here we don't need it. (It isn't particularly hard either.) In this example *the limiting value is easy to identify*. The average $2t + h$ approaches $2t$, as $h \to 0$.

What remains to do in this section? We answered Question 2—to find velocity from distance. We have not answered Question 1. If $v(t) = 2t$ increases linearly with time, what is the distance? This goes in the opposite direction (it is *integration*).

The Fundamental Theorem of Calculus says that no new work is necessary. *If the slope of $f(t)$ leads to $v(t)$, then the area under that v-graph leads back to the f-graph*. The odometer readings $f = t^2$ produced speedometer readings $v = 2t$. By the Fundamental Theorem, the area under $2t$ should be t^2. But we have certainly not proved any fundamental theorems, so it is better to be safe—by actually computing the area.

Fortunately, it is the area of a triangle. The base of the triangle is t and the height is $v = 2t$. The area agrees with $f(t)$:

$$\text{area} = \tfrac{1}{2}(\text{base})(\text{height}) = \tfrac{1}{2}(t)(2t) = t^2. \tag{4}$$

EXAMPLE 1 The graphs are *shifted in time*. The car doesn't start until $t = 1$. Therefore $v = 0$ and $f = 0$ up to that time. After the car starts we have $v = 2(t - 1)$ and $f = (t - 1)^2$. You see how the time delay of 1 enters the formulas. Figure 1.13 shows how it affects the graphs.
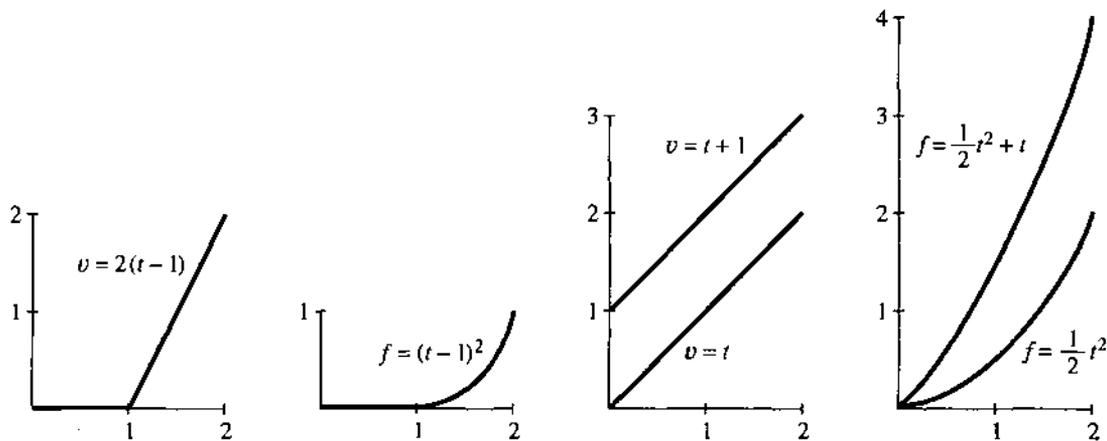
Fig. 1.13 Delayed velocity and distance. The pairs $v = at + b$ and $f = \tfrac{1}{2}at^2 + bt$.

EXAMPLE 2 The acceleration changes from 2 to another constant a. The velocity changes from $v = 2t$ to $v = at$. *The acceleration is the slope of the velocity curve!* The distance is also proportional to a, but notice the factor $\tfrac{1}{2}$:

$$\text{acceleration } a \quad \Leftrightarrow \quad \text{velocity } v = at \quad \Leftrightarrow \quad \text{distance } f = \tfrac{1}{2}at^2.$$

If a equals 1, then $v = t$ and $f = \tfrac{1}{2}t^2$. That is one of the most famous pairs in calculus. If a equals the gravitational constant g, then $v = gt$ is the velocity of a falling body. The speed doesn't depend on the mass (tested by Galileo at the Leaning Tower of Pisa). Maybe he saw the distance $f = \tfrac{1}{2}gt^2$ more easily than the speed $v = gt$. Anyway, this is the most famous pair in physics.

EXAMPLE 3 Suppose $f(t) = 3t + t^2$. The average velocity from t to $t + h$ is

$$v_{ave} = \frac{f(t + h) - f(t)}{h} = \frac{3(t + h) + (t + h)^2 - 3t - t^2}{h}.$$

The change in distance has an extra $3h$ (coming from $3(t + h)$ minus $3t$). The velocity contains an additional 3 (coming from $3h$ divided by h). When $3t$ is added to the distance, 3 is added to the velocity. If Galileo had thrown a weight instead of dropping it, the starting velocity v_0 would have added $v_0 t$ to the distance.

FUNCTIONS ACROSS TIME

The idea of slope is not difficult—for one straight line. Divide the change in f by the change in t. In Chapter 2, divide the change in y by the change in x. Experience shows that the hard part is to see what happens to the slope as the line moves.

Figure 1.14a shows the line between points A and B on the curve. This is a "secant line." Its slope is an *average* velocity. What calculus does is to bring that point B *down the curve toward A*.
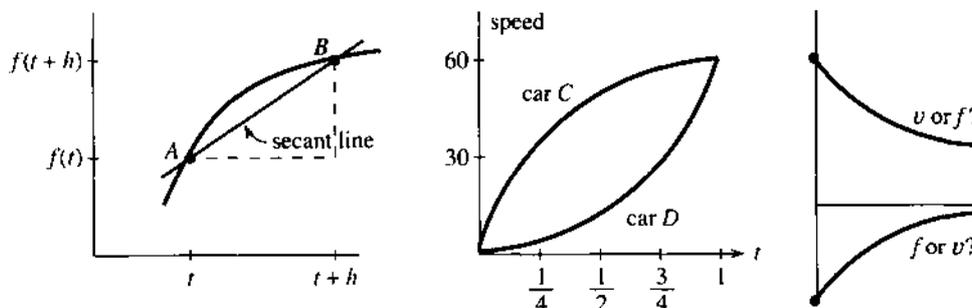
Fig. 1.14   Slope of line, slope of curve. Two velocity graphs. *Which is which?*

Question 1   What happens to the "change in f"—the height of B above A?
Answer   The change in f decreases to zero. So does the change in t.

Question 2   As B approaches A, does the slope of the line increase or decrease?
Answer   I am not going to answer that question. It is too important. Draw another secant line with B closer to A. Compare the slopes.

This question was created by Steve Monk at the University of Washington—where 57% of the class gave the right answer. Probably 97% would have found the right slope from a formula. Figure 1.14b shows the opposite problem. We know the velocity, not the distance. But calculus answers questions about both functions.

Question 3   Which car is going faster at time $t = 3/4$?
Answer   Car C has higher speed. Car D has greater acceleration.

Question 4   If the cars start together, is D catching up to C at the end? Between $t = \frac{1}{2}$ and $t = 1$, do the cars get closer or further apart?
Answer   This time more than half the class got it wrong. You won't but you can see why they did. You have to look at the speed graph and imagine the distance graph. When car C is going faster, the distance between them _____.

To repeat: The cars start together, but they don't finish together. They reach the same speed at $t = 1$, not the same distance. Car C went faster. You really should draw their distance graphs, to see how they bend.

These problems help to emphasize one more point. Finding the speed (or slope) is entirely different from finding the distance (or area):

1. To find the *slope* of the f-graph at a particular time t, you *don't* have to know the whole history.
2. To find the *area* under the v-graph up to a particular time t, you *do* have to know the whole history.

A short record of distance is enough to recover $v(t)$. Point B moves toward point A. The problem of slope is *local*—the speed is completely decided by $f(t)$ near point A.

In contrast, a short record of speed is *not enough* to recover the total distance. We have to know what the mileage was earlier. Otherwise we can only know the *increase* in mileage, not the total.

1.3 EXERCISES

Read-through questions

Between the distances $f(2) = 100$ and $f(6) = 200$, the average velocity is ___a___. If $f(t) = \frac{1}{4}t^2$ then $f(6) =$ ___b___ and $f(8) =$ ___c___. The average velocity in between is ___d___. The instantaneous velocities at $t = 6$ and $t = 8$ are ___e___ and ___f___.

The average velocity is computed from $f(t)$ and $f(t + h)$ by $v_{ave} =$ ___g___. If $f(t) = t^2$ then $v_{ave} =$ ___h___. From $t = 1$ to $t = 1.1$ the average is ___i___. The instantaneous velocity is the ___j___ of v_{ave}. If the distance is $f(t) = \frac{1}{2}at^2$ then the velocity is $v(t) =$ ___k___ and the acceleration is ___l___.

On the graph of $f(t)$, the average velocity between A and B is the slope of ___m___. The velocity at A is found by ___n___. The velocity at B is found by ___o___. When the velocity is positive, the distance is ___p___. When the velocity is increasing, the car is ___q___.

1 Compute the average velocity between $t = 5$ and $t = 8$:

(a) $f(t) = 6t$ (b) $f(t) = 6t + 2$
(c) $f(t) = \frac{1}{2}at^2$ (d) $f(t) = t - t^2$
(e) $f(t) = 6$ (f) $v(t) = 2t$

2 For the same functions compute $[f(t + h) - f(t)]/h$. This depends on t and h. Find the limit as $h \to 0$.

3 If the odometer reads $f(t) = t^2 + t$ (f in miles or kilometers, t in hours), find the average speed between

(a) $t = 1$ and $t = 2$
(b) $t = 1$ and $t = 1.1$
(c) $t = 1$ and $t = 1 + h$
(d) $t = 1$ and $t = .9$ (note $h = -.1$)

4 For the same $f(t) = t^2 + t$, find the average speed between

(a) $t = 0$ and 1 (b) $t = 0$ and $\frac{1}{2}$ (c) $t = 0$ and h.

5 In the answer to 3(c), find the limit as $h \to 0$. What does that limit tell us?

6 Set $h = 0$ in your answer to 4(c). Draw the graph of $f(t) = t^2 + t$ and show its slope at $t = 0$.

7 Draw the graph of $v(t) = 1 + 2t$. From geometry find the area under it from 0 to t. Find the slope of that area function $f(t)$.

8 Draw the graphs of $v(t) = 3 - 2t$ and the area $f(t)$.

9 *True or false*

(a) If the distance $f(t)$ is positive, so is $v(t)$.
(b) If the distance $f(t)$ is increasing, so is $v(t)$.
(c) If $f(t)$ is positive, $v(t)$ is increasing.
(d) If $v(t)$ is positive, $f(t)$ is increasing.

10 If $f(t) = 6t^2$ find the slope of the f-graph and also the v-graph. The slope of the v-graph is the _____.

11 If $f(t) = t^2$ what is the average velocity between $t = .9$ and $t = 1.1$? What is the average between $t - h$ and $t + h$?

12 (a) Show that for $f(t) = \frac{1}{2}at^2$ the average velocity between $t - h$ and $t + h$ is exactly the velocity at t.

(b) The area under $v(t) = at$ from $t - h$ to $t + h$ is exactly the base $2h$ times _____.

13 Find $f(t)$ from $v(t) = 20t$ if $f(0) = 12$. Also if $f(1) = 12$.

14 *True or false*, for any distance curves.

(a) The slope of the line from A to B is the average velocity between those points.

(b) Secant lines have smaller slopes than the curve.

(c) If $f(t)$ and $F(t)$ start together and finish together, the average velocities are equal.

(d) If $v(t)$ and $V(t)$ start together and finish together, the increases in distance are equal.

15 When you jump up and fall back your height is $y = 2t - t^2$ in the right units.

(a) Graph this parabola and its slope.

(b) Find the time in the air and maximum height.

(c) *Prove: Half the time you are above $y = \frac{3}{4}$.*

Basketball players "hang" in the air partly because of (c).

16 Graph $f(t) = t^2$ and $g(t) = f(t) - 2$ and $h(t) = f(2t)$, all from $t = 0$ to $t = 1$. Find the velocities.

17 (Recommended) An up and down velocity is $v(t) = 2t$ for $t \leqslant 3$, $v(t) = 12 - 2t$ for $t \geqslant 3$. Draw the piecewise parabola $f(t)$. Check that $f(6)$ = area under the graph of $v(t)$.

18 Suppose $v(t) = t$ for $t \leqslant 2$ and $v(t) = 2$ for $t \geqslant 2$. Draw the graph of $f(t)$ out to $t = 3$.

19 Draw $f(t)$ up to $t = 4$ when $v(t)$ increases linearly from

(a) 0 to 2   (b) -1 to 1   (c) -2 to 0.

20 (Recommended) Suppose $v(t)$ is the piecewise linear sine function of Section 1.2. (In Figure 1.8 it was the distance.)

Find the area under $v(t)$ between $t = 0$ and $t = 1, 2, 3, 4, 5, 6$. Plot those points $f(1), \dots, f(6)$ and draw the complete piecewise parabola $f(t)$.

21 Draw the graph of $f(t) = |1 - t^2|$ for $0 \leqslant t \leqslant 2$. Find a three-part formula for $v(t)$.

22 Draw the graphs of $f(t)$ for these velocities (to $t = 2$):

(a) $v(t) = 1 - t$

(b) $v(t) = |1 - t|$

(c) $v(t) = (1 - t) + |1 - t|$.

23 When does $f(t) = t^2 - 3t$ reach 10? Find the average velocity up to that time and the instantaneous velocity at that time.

24 If $f(t) = \frac{1}{2}at^2 + bt + c$, what is $v(t)$? What is the slope of $v(t)$? When does $f(t)$ equal 41, if $a = b = c = 1$?

25 If $f(t) = t^2$ then $v(t) = 2t$. Does the speeded-up function $f(4t)$ have velocity $v(4t)$ or $4v(t)$ or $4v(4t)$?

26 If $f(t) = t - t^2$ find $v(t)$ and $f(3t)$. Does the slope of $f(3t)$ equal $v(3t)$ or $3v(t)$ or $3v(3t)$?

27 For $f(t) = t^2$ find $v_{ave}(t)$ between 0 and t. Graph $v_{ave}(t)$ and $v(t)$.

28 If you know the average velocity $v_{ave}(t)$, how can you find the distance $f(t)$? Start from $f(0) = 0$.

1.4 Circular Motion

This section introduces completely new distances and velocities—*the sines and cosines from trigonometry*. As I write that last word, I ask myself how much trigonometry it is essential to know. There will be the basic picture of a right triangle, with sides cos t and sin t and 1. There will also be the crucial equation $(\cos t)^2 + (\sin t)^2 = 1$, which is Pythagoras' law $a^2 + b^2 = c^2$. The squares of two sides add to the square of the hypotenuse (and the 1 is really 1^2). Nothing else is needed immediately. If you don't know trigonometry, don't stop—an important part can be learned now.

You will recognize the wavy graphs of the sine and cosine. *We intend to find the slopes of those graphs.* That can be done without using the formulas for $\sin(x + y)$ and $\cos(x + y)$—which later give the same slopes in a more algebraic way. Here it is only basic things that are needed.† And anyway, how complicated can a triangle be?

Remark You might think trigonometry is only for surveyors and navigators (people with triangles). Not at all! By far the biggest applications are to *rotation* and *vibration* and *oscillation*. It is fantastic that sines and cosines are so perfect for "repeating motion"—around a circle or up and down.

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†Sines and cosines are so important that I added a review of trigonometry in Section 1.5. But the concepts in this section can be more valuable than formulas.
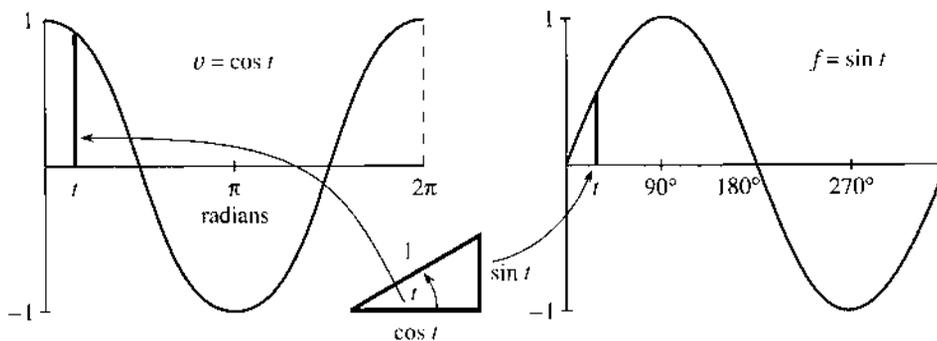
Fig. 1.15 As the angle t changes, the graphs show the sides of the right triangle.

Our underlying goal is to offer one more example in which the velocity can be computed by common sense. Calculus is mainly an extension of common sense, but here that extension is not needed. We will find the slope of the sine curve. The straight line $f = vt$ was easy and the parabola $f = \frac{1}{2}at^2$ was harder. The new example also involves realistic motion, seen every day. We start with **circular motion**, in which the position is given and the velocity will be found.

A ball goes around a circle of radius one. The center is at $x = 0$, $y = 0$ (the origin). The x and y coordinates satisfy $x^2 + y^2 = 1^2$, to keep the ball on the circle. We specify its position in Figure 1.16a by giving its angle with the horizontal. And we make the ball travel with constant speed, by requiring that *the angle is equal to the time t.* The ball goes counterclockwise. At time 1 it reaches the point where the angle equals 1. The angle is measured in **radians** rather than degrees, so a full circle is completed at $t = 2\pi$ instead of $t = 360$.

The ball starts on the x axis, where the angle is zero. Now find it at time t:

The ball is at the point where $x = \cos t$ *and* $y = \sin t$.

This is where trigonometry is useful. The cosine oscillates between 1 and -1, as the ball goes from far right to far left and back again. The sine also oscillates between 1 and -1, starting from $\sin 0 = 0$. At time $\pi/2$ the sine (the height) increases to one. The cosine is zero and the ball reaches the top point $x = 0$, $y = 1$. At time π the cosine is -1 and the sine is back to zero—the coordinates are $(-1, 0)$. At $t = 2\pi$ the circle is complete (the angle is also 2π), and $x = \cos 2\pi = 1$, $y = \sin 2\pi = 0$.
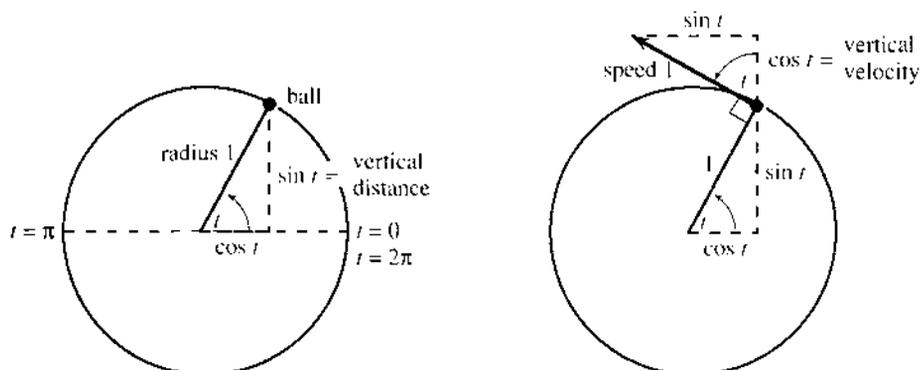
Fig. 1.16 Circular motion with speed 1, angle t, height $\sin t$, upward velocity $\cos t$.

Important point: The distance around the circle (its circumference) is $2\pi r = 2\pi$, because the radius is 1. The ball travels a distance 2π in a time 2π. *The speed equals 1.* It remains to find the velocity, which involves not only speed but *direction.*

Degrees vs. radians   A full circle is 360 degrees and 2π radians. Therefore

$$1 \text{ radian} = 360/2\pi \text{ degrees} \approx 57.3 \text{ degrees}$$

$$1 \text{ degree} = 2\pi/360 \text{ radians} \approx .01745 \text{ radians}$$

Radians were invented to avoid those numbers! The speed is exactly 1, reaching t radians at time t. The speed would be .01745, if the ball only reached t degrees. The ball would complete the circle at time $T = 360$. We cannot accept the division of the circle into 360 pieces (by whom?), which produces these numbers.

To check degree mode vs. radian mode, verify that $\sin 1° \approx .017$ and $\sin 1 \approx .84$.

VELOCITY OF THE BALL

At time t, which direction is the ball going? Calculus watches the motion between t and $t + h$. For a ball on a string, we don't need calculus—just let go. *The direction of motion is tangent to the circle.* With no force to keep it on the circle, *the ball goes off on a tangent.* If the ball is the moon, the force is gravity. If it is a hammer swinging around on a chain, the force is from the center. When the thrower lets go, the hammer takes off—and it is an art to pick the right moment. (I once saw a friend hit by a hammer at MIT. He survived, but the thrower quit track.) Calculus will find that same tangent direction, when the points at t and $t + h$ come close.

The "*velocity triangle*" is in Figure 1.16b. It is the same as the position triangle, but rotated through 90°. The hypotenuse is tangent to the circle, in the direction the ball is moving. Its length equals 1 (the speed). The angle t still appears, but now it is the angle with the vertical. *The upward component of velocity is $\cos t$, when the upward component of position is $\sin t$.* That is our common sense calculation, based on a figure rather than a formula. The rest of this section depends on it—and we check $v = \cos t$ at special points.

At the starting time $t = 0$, the movement is all upward. The height is $\sin 0 = 0$ and the upward velocity is $\cos 0 = 1$. At time $\pi/2$, the ball reaches the top. The height is $\sin \pi/2 = 1$ and the upward velocity is $\cos \pi/2 = 0$. At that instant the ball is not moving up or down.

The horizontal velocity contains a minus sign. At first the ball travels to the *left*. The value of x is $\cos t$, but *the speed in the x direction is* $-\sin t$. Half of trigonometry is in that figure (the good half), and you see how $\sin^2 t + \cos^2 t = 1$ is so basic. That equation applies to position and velocity, at every time.

Application of plane geometry: The right triangles in Figure 1.16 are the same size and shape. They look congruent and they are—the angle t above the ball equals the angle t at the center. That is because the three angles at the ball add to 180°.

OSCILLATION: UP AND DOWN MOTION

We now use circular motion to study *straight-line motion.* That line will be the y axis. Instead of a ball going around a circle, a mass will move up and down. It oscillates between $y = 1$ and $y = -1$. *The mass is the "shadow of the ball,"* as we explain in a moment.

There is a jumpy oscillation that we do not want, with $v = 1$ and $v = -1$. That "bang-bang" velocity is like a billiard ball, bouncing between two walls without slowing down. If the distance between the walls is 2, then at $t = 4$ the ball is back to the start. The distance graph is a zigzag (or sawtooth) from Section 1.2.

We prefer a smoother motion. Instead of velocities that jump between $+1$ and -1, a real oscillation *slows down to zero* and gradually builds up speed again. The mass is on a spring, which pulls it back. The velocity drops to zero as the spring is fully stretched. Then v is negative, as the mass goes the same distance in the opposite direction. *Simple harmonic motion* is the most important back and forth motion, while $f = vt$ and $f = \frac{1}{2}at^2$ are the most important one-way motions.
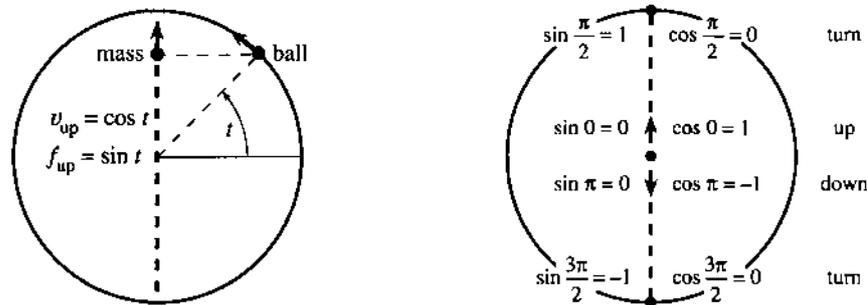
Fig. 1.17 Circular motion of the ball and harmonic motion of the mass (its shadow).

How do we describe this oscillation? The best way is to match it with the ball on the circle. *The height of the ball will be the height of the mass.* The "shadow of the ball" goes up and down, level with the ball. As the ball passes the top of the circle, the mass stops at the top and starts down. As the ball goes around the bottom, the mass stops and turns back up the y axis. Halfway up (or down), the speed is 1.

Figure 1.17a shows the mass at a typical time t. The height is $y = f(t) = \sin t$, level with the ball. This height oscillates between $f = 1$ and $f = -1$. But the mass does not move with constant speed. *The speed of the mass is changing although the speed of the ball is always* 1. The time for a full cycle is still 2π, but within that cycle the mass speeds up and slows down. The problem is to find the changing velocity v. Since the distance is $f = \sin t$, the velocity will be the *slope of the sine curve*.

THE SLOPE OF THE SINE CURVE

At the top and bottom ($t = \pi/2$ and $t = 3\pi/2$) the ball changes direction and $v = 0$. *The slope at the top and bottom of the sine curve is zero.*† At time zero, when the ball is going straight up, the slope of the sine curve is $v = 1$. At $t = \pi$, when the ball and mass and f-graph are going down, the velocity is $v = -1$. The mass goes fastest at the center. The mass goes slowest (in fact it stops) when the height reaches a maximum or minimum. The velocity triangle yields v at every time t.

To find the upward velocity of the mass, look at the upward velocity of the ball. Those velocities are the same! The mass and ball stay level, and we know v from circular motion: *The upward velocity is* $v = \cos t$.

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†That looks easy but you will see later that it is extremely important. *At a maximum or minimum the slope is zero.* The curve levels off.

Figure 1.18 shows the result we want. On the right, $f = \sin t$ gives the height. On the left is the velocity $v = \cos t$. That velocity is the slope of the f-curve. The height and velocity (red lines) are oscillating together, but they are out of phase—just as the position triangle and velocity triangle were at right angles. This is absolutely fantastic, that in calculus the two most famous functions of trigonometry form a pair: *The slope of the sine curve is given by the cosine curve.*

When the distance is $f(t) = \sin t$, the velocity is $v(t) = \cos t$.

Admission of guilt: The slope of $\sin t$ was not computed in the standard way. Previously we compared $(t + h)^2$ with t^2, and divided that distance by h. This average velocity approached the slope $2t$ as h became small. *For $\sin t$ we could have done the same*:

$$\text{average velocity} = \frac{\text{change in } \sin t}{\text{change in } t} = \frac{\sin(t + h) - \sin t}{h}. \tag{1}$$

This is where we need the formula for $\sin(t + h)$, coming soon. Somehow the ratio in (1) should approach $\cos t$ as $h \to 0$. (It does.) The sine and cosine fit the same pattern as t^2 and $2t$—our shortcut was to watch the shadow of motion around a circle.
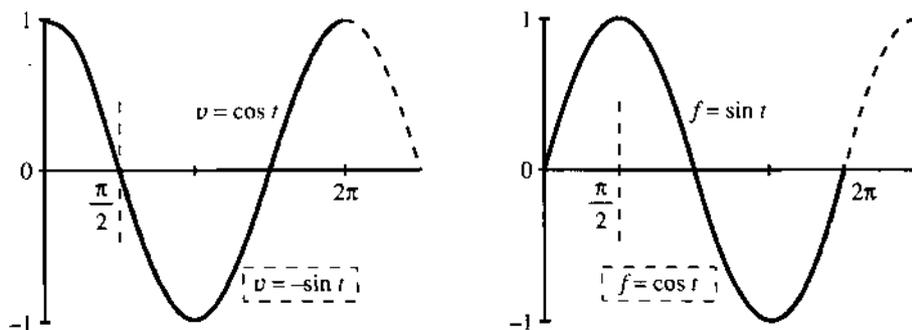
Fig. 1.18 $v = \cos t$ when $f = \sin t$ (red); $v = -\sin t$ when $f = \cos t$ (black).

Question 1 *What if the ball goes twice as fast, to reach angle $2t$ at time t?*

Answer The speed is now 2. The time for a full circle is only π. The ball's position is $x = \cos 2t$ and $y = \sin 2t$. The velocity is still tangent to the circle—but the tangent is at angle $2t$ where the ball is. Therefore $\cos 2t$ enters the upward velocity and $-\sin 2t$ enters the horizontal velocity. The difference is that *the velocity triangle is twice as big*. The upward velocity is not $\cos 2t$ but $2\cos 2t$. The horizontal velocity is $-2\sin 2t$. Notice these 2's!

Question 2 *What is the area under the cosine curve from $t = 0$ to $t = \pi/2$?*

You can answer that, if you accept the Fundamental Theorem of Calculus— *computing areas is the opposite of computing slopes*. The slope of $\sin t$ is $\cos t$, so the area under $\cos t$ is the increase in $\sin t$. No reason to believe that yet, but we use it anyway.

From $\sin 0 = 0$ to $\sin \pi/2 = 1$, the increase is 1. Please realize the power of calculus. No other method could compute the area under a cosine curve so fast.

THE SLOPE OF THE COSINE CURVE

I cannot resist uncovering another distance and velocity (another f-v pair) with no extra work. This time f is the cosine. The time clock starts *at the top of the circle*. The old time $t = \pi/2$ is now $t = 0$. The dotted lines in Figure 1.18 show the new start. But the shadow has exactly the same motion—the ball keeps going around the circle, and the mass follows it up and down. The f-graph and v-graph are still correct, both with a time shift of $\pi/2$.

The new f-graph is the cosine. The new v-graph is **minus the sine**. The slope of the cosine curve follows the *negative* of the sine curve. That is another famous pair, twins of the first:

When the distance is $f(t) = \cos t$, **the velocity is** $v(t) = -\sin t$.

You could see that coming, by watching the ball go left and right (instead of up and down). Its distance across is $f = \cos t$. Its velocity across is $v = -\sin t$. That twin pair completes the calculus in Chapter 1 (trigonometry to come). We review the ideas:

> v is the *velocity*
>> the *slope* of the distance curve
>> the *limit* of average velocity over a short time
>> the *derivative* of f.
>
> f is the *distance*
>> the *area* under the velocity curve
>> the *limit* of total distance over many short times
>> the *integral* of v.

Differential calculus: Compute v from f.   *Integral calculus*: Compute f from v.

With constant velocity, f equals vt. With constant acceleration, $v = at$ and $f = \frac{1}{2}at^2$. In harmonic motion, $v = \cos t$ and $f = \sin t$. One part of our goal is to extend that list—for which we need the tools of calculus. Another and more important part is to put these ideas to use.

Before the chapter ends, may I add a note about the book and the course? The book is more personal than usual, and I hope readers will approve. What I write is very close to what I would say, if you were in this room. The sentences are spoken before they are written.† Calculus is alive and moving forward—it needs to be taught that way.

One new part of the subject has come with the computer. It works with a finite step h, not an "infinitesimal" limit. What it can do, it does quickly—even if it cannot find exact slopes or areas. The result is an overwhelming growth in the range of problems that can be solved. We landed on the moon because f and v were so accurate. (The moon's orbit has sines and cosines, the spacecraft starts with $v = at$ and $f = \frac{1}{2}at^2$. Only the computer can account for the atmosphere and the sun's gravity and the changing mass of the spacecraft.) *Modern mathematics is a combination of exact formulas and approximate computations*. Neither part can be ignored, and I hope you will see numerically what we derive algebraically. The exercises are to help you master both parts.

---

†On television you know immediately when the words are live. The same with writing.

The course has made a quick start—not with an abstract discussion of sets or functions or limits, but with the concrete questions that led to those ideas. You have seen a distance function f and a limit v of average velocities. We will meet more functions and more limits (and their definitions!) but it is crucial to study important examples early. There is a lot to do, but the course has definitely begun.

1.4 EXERCISES

Read-through questions

A ball at angle t on the unit circle has coordinates $x =$ __a__ and $y =$ __b__. It completes a full circle at $t =$ __c__. Its speed is __d__. Its velocity points in the direction of the __e__, which is __f__ to the radius coming out from the center. The upward velocity is __g__ and the horizontal velocity is __h__.

A mass going up and down level with the ball has height $f(t) =$ __i__. This is called simple __j__ motion. The velocity is $v(t) =$ __k__. When $t = \pi/2$ the height is $f =$ __l__ and the velocity is $v =$ __m__. If a speeded-up mass reaches $f = \sin 2t$ at time t, its velocity is $v =$ __n__. A shadow traveling *under* the ball has $f = \cos t$ and $v =$ __o__. When f is distance = area = integral, v is __p__ = __q__ = __r__.

1 For a ball going around a unit circle with speed 1,

  (a) how long does it take for 5 revolutions?

  (b) at time $t = 3\pi/2$ where is the ball?

  (c) at $t = 22$ where is the ball (approximately)?

2 For the same motion find the exact x and y coordinates at $t = 2\pi/3$. At what time would the ball hit the x axis, if it goes off on the tangent at $t = 2\pi/3$?

3 A ball goes around a circle of radius 4. At time t (when it reaches angle t) find

  (a) its x and y coordinates

  (b) the speed and the distance traveled

  (c) the vertical and horizontal velocity.

4 On a circle of radius R find the x and y coordinates at time t (and angle t). Draw the velocity triangle and find the x and y velocities.

5 A ball travels around a unit circle (radius 1) with speed 3, starting from angle zero. At time t,

  (a) what angle does it reach?

  (b) what are its x and y coordinates?

  (c) what are its x and y velocities? This part is harder.

6 If another ball stays $\pi/2$ radians ahead of the ball with speed 3, find its angle, its x and y coordinates, and its vertical velocity at time t.

7 A mass moves on the x axis under or over the original ball (on the unit circle with speed 1). What is the position $x = f(t)$? Find x and v at $t = \pi/4$. Plot x and v up to $t = \pi$.

8 Does the new mass (under or over the ball) meet the old mass (level with the ball)? What is the distance between the masses at time t?

9 Draw graphs of $f(t) = \cos 3t$ and $\cos 2\pi t$ and $2\pi \cos t$, marking the time axes. How long until each f repeats?

10 Draw graphs of $f = \sin(t + \pi)$ and $v = \cos(t + \pi)$. This oscillation stays level with what ball?

11 Draw graphs of $f = \sin(\pi/2 - t)$ and $v = -\cos(\pi/2 - t)$. This oscillation stays level with a ball going which way starting where?

12 Draw a graph of $f(t) = \sin t + \cos t$. Estimate its greatest height (maximum f) and the time it reaches that height. By computing f^2 check your estimate.

13 How fast should you run across the circle to meet the ball again? It travels at speed 1.

14 A mass falls from the top of the unit circle when the ball of speed 1 passes by. What acceleration a is necessary to meet the ball at the bottom?

Find the area under $v = \cos t$ from the change in $f = \sin t$:

15 from $t = 0$ to $t = \pi$

16 from $t = 0$ to $t = \pi/6$

17 from $t = 0$ to $t = 2\pi$

18 from $t = \pi/2$ to $t = 3\pi/2$.

19 The distance curve $f = \sin 4t$ yields the velocity curve $v = 4 \cos 4t$. Explain both 4's.

20 The distance curve $f = 2 \cos 3t$ yields the velocity curve $v = -6 \sin 3t$. Explain the -6.

21 The velocity curve $v = \cos 4t$ yields the distance curve $f = \frac{1}{4} \sin 4t$. Explain the $\frac{1}{4}$.

22 The velocity $v = 5 \sin 5t$ yields what distance?

23 Find the slope of the sine curve at $t = \pi/3$ from $v = \cos t$. Then find an average slope by dividing $\sin \pi/2 - \sin \pi/3$ by the time difference $\pi/2 - \pi/3$.

24 The slope of $f = \sin t$ at $t = 0$ is $\cos 0 = 1$. Compute average slopes $(\sin t)/t$ for $t = 1, .1, .01, .001$.

The ball at $x = \cos t$, $y = \sin t$ **circles (1) counterclockwise (2) with radius 1 (3) starting from** $x = 1$, $y = 0$ **(4) at speed 1. Find (1)(2)(3)(4) for the motions 25–30.**

25 $x = \cos 3t$, $y = -\sin 3t$

26 $x = 3 \cos 4t$, $y = 3 \sin 4t$

27 $x = 5 \sin 2t$, $y = 5 \cos 2t$

28 $x = 1 + \cos t$, $y = \sin t$

29 $x = \cos(t + 1)$, $y = \sin(t + 1)$

30 $x = \cos(-t)$, $y = \sin(-t)$

The oscillation $x = 0$, $y = \sin t$ **goes (1) up and down (2) between** -1 **and** 1 **(3) starting from** $x = 0$, $y = 0$ **(4) at velocity** $v = \cos t$. **Find (1)(2)(3)(4) for the oscillations 31–36.**

31 $x = \cos t$, $y = 0$ **32** $x = 0$, $y = \sin 5t$

33 $x = 0$, $y = 2\sin(t + \theta)$ **34** $x = \cos t$, $y = \cos t$

35 $x = 0$, $y = -2\cos \frac{1}{2}t$ **36** $x = \cos^2 t$, $y = \sin^2 t$

37 If the ball on the unit circle reaches t *degrees* at time t, find its position and speed and upward velocity.

38 Choose the number k so that $x = \cos kt$, $y = \sin kt$ completes a rotation at $t = 1$. Find the speed and upward velocity.

39 If a pitcher doesn't pause before starting to throw, a balk is called. The American League decided mathematically that there is always a stop between backward and forward motion, even if the time is too short to see it. (Therefore no balk.) Is that true?

1.5 A Review of Trigonometry

Trigonometry begins with a right triangle. The size of the triangle is not as important as the angles. We focus on one particular angle—call it θ—and on the *ratios* between the three sides x, y, r. The ratios don't change if the triangle is scaled to another size. Three sides give six ratios, which are the basic functions of trigonometry:
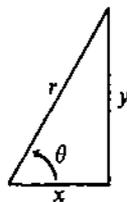
$$\cos \theta = \frac{x}{r} = \frac{\text{near side}}{\text{hypotenuse}} \qquad \sec \theta = \frac{r}{x} = \frac{1}{\cos \theta}$$

$$\sin \theta = \frac{y}{r} = \frac{\text{opposite side}}{\text{hypotenuse}} \qquad \csc \theta = \frac{r}{y} = \frac{1}{\sin \theta}$$

$$\tan \theta = \frac{y}{x} = \frac{\text{opposite side}}{\text{near side}} \qquad \cot \theta = \frac{x}{y} = \frac{1}{\tan \theta}$$

Fig. 1.19

Of course those six ratios are not independent. The three on the right come directly from the three on the left. And the tangent is the sine divided by the cosine:

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{y/r}{x/r} = \frac{y}{x}.$$

Note that "tangent of an angle" and "tangent to a circle" and "tangent line to a graph" are different uses of the same word. As the cosine of θ goes to zero, the tangent of θ goes to infinity. The side x becomes zero, θ approaches $90°$, and the triangle is infinitely steep. The sine of $90°$ is $y/r = 1$.

Triangles have a serious limitation. They are excellent for angles up to $90°$, and they are OK up to $180°$, but after that they fail. We cannot put a $240°$ angle into a triangle. Therefore we change now to a circle.
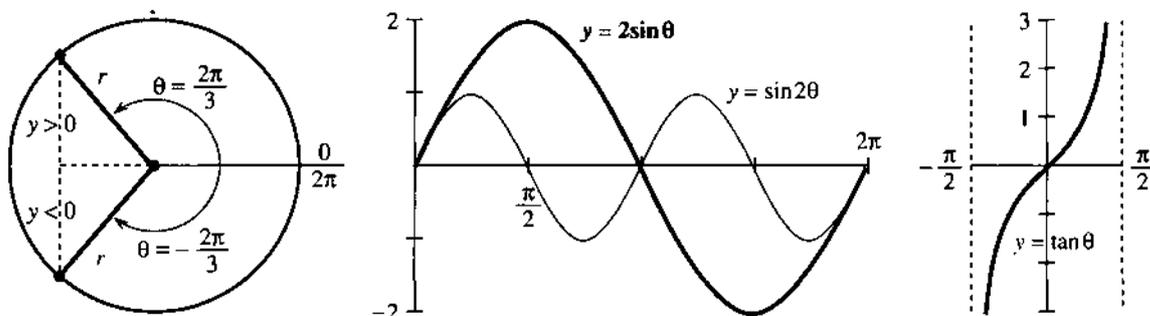
Fig. 1.20 Trigonometry on a circle. Compare $2 \sin \theta$ with $\sin 2\theta$ and $\tan \theta$ (periods 2π, π, π).

Angles are measured from the positive x axis (counterclockwise). Thus $90°$ is straight up, $180°$ is to the left, and $360°$ is in the same direction as $0°$. (Then $450°$ is the same as $90°$.) Each angle yields a point on the circle of radius r. The coordinates x and y of that point can be negative (*but never r*). As the point goes around the circle, the six ratios $\cos \theta$, $\sin \theta$, $\tan \theta$, ... trace out six graphs. The cosine waveform is the same as the sine waveform—just shifted by $90°$.

One more change comes with the move to a circle. Degrees are out. Radians are in. The distance around the whole circle is $2\pi r$. The distance around to other points is θr. *We measure the angle by that multiple θ.* For a half-circle the distance is πr, so the angle is π radians—which is $180°$. A quarter-circle is $\pi/2$ radians or $90°$. *The distance around to angle θ is r times θ.*

When $r = 1$ this is the ultimate in simplicity: *The distance is θ.* A $45°$ angle is $\frac{1}{8}$ of a circle and $2\pi/8$ radians—and the length of the circular arc is $2\pi/8$. Similarly for $1°$:

$$360° = 2\pi \text{ radians} \qquad 1° = 2\pi/360 \text{ radians} \qquad 1 \text{ radian} = 360/2\pi \text{ degrees.}$$

An angle going clockwise is *negative*. The angle $-\pi/3$ is $-60°$ and takes us $\frac{1}{6}$ of the *wrong* way around the circle. What is the effect on the six functions?

Certainly the radius r is not changed when we go to $-\theta$. Also x is not changed (see Figure 1.20a). But y reverses sign, because $-\theta$ is below the axis when $+\theta$ is above. This change in y affects y/r and y/x but not x/r:

$$\cos(-\theta) = \cos \theta \qquad \sin(-\theta) = -\sin \theta \qquad \tan(-\theta) = -\tan \theta.$$

The cosine is *even* (no change). The sine and tangent are *odd* (change sign).

The same point is $\frac{5}{6}$ of the *right* way around. Therefore $\frac{5}{6}$ of 2π radians (or $300°$) gives the same direction as $-\pi/3$ radians or $-60°$. *A difference of 2π makes no difference to x, y, r.* Thus $\sin \theta$ and $\cos \theta$ and the other four functions have period 2π. We can go five times or a hundred times around the circle, adding 10π or 200π to the angle, and the six functions repeat themselves.

EXAMPLE Evaluate the six trigonometric functions at $\theta = 2\pi/3$ (or $\theta = -4\pi/3$).

This angle is shown in Figure 1.20a (where $r = 1$). The ratios are

$$\cos \theta = x/r = -1/2 \qquad \sin \theta = y/r = \sqrt{3}/2 \qquad \tan \theta = y/x = -\sqrt{3}$$

$$\sec \theta = -2 \qquad \csc \theta = 2/\sqrt{3} \qquad \cot \theta = -1/\sqrt{3}$$

Those numbers illustrate basic facts about the sizes of four functions:

$$|\cos \theta| \leqslant 1 \qquad |\sin \theta| \leqslant 1 \qquad |\sec \theta| \geqslant 1 \qquad |\csc \theta| \geqslant 1.$$

The tangent and cotangent can fall anywhere, as long as $\cot \theta = 1/\tan \theta$.

The numbers reveal more. The tangent $-\sqrt{3}$ is the ratio of sine to cosine. The secant -2 is $1/\cos \theta$. Their squares are 3 and 4 (differing by 1). That may not seem remarkable, but it is. There are three relationships in the *squares* of those six numbers, and they are the key identities of trigonometry:

$$\cos^2 \theta + \sin^2 \theta = 1 \qquad 1 + \tan^2 \theta = \sec^2 \theta \qquad \cot^2 \theta + 1 = \csc^2 \theta$$

Everything flows from the Pythagoras formula $x^2 + y^2 = r^2$. Dividing by r^2 gives $(x/r)^2 + (y/r)^2 = 1$. That is $\cos^2 \theta + \sin^2 \theta = 1$. Dividing by x^2 gives the second identity, which is $1 + (y/x)^2 = (r/x)^2$. Dividing by y^2 gives the third. All three will be needed throughout the book—and the first one has to be unforgettable.

DISTANCES AND ADDITION FORMULAS

To compute the distance between points we stay with Pythagoras. The points are in Figure 1.21a. They are known by their x and y coordinates, and d is the distance between them. The third point completes a right triangle.

For the x distance along the bottom we don't need help. It is $x_2 - x_1$ (or $|x_2 - x_1|$ since distances can't be negative). The distance up the side is $|y_2 - y_1|$. Pythagoras immediately gives the distance d:

$$\text{\textit{distance between points}} = d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \tag{1}$$
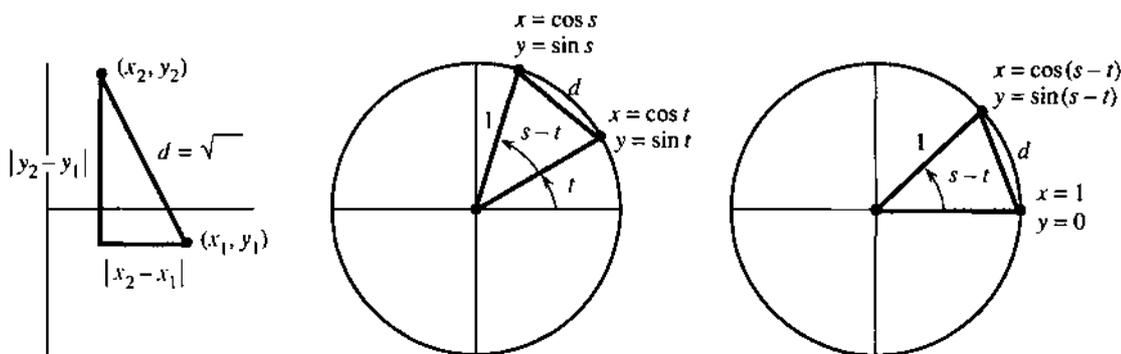
Fig. 1.21 Distance between points and equal distances in two circles.

By applying this distance formula in two identical circles, we discover the cosine of $s - t$. (Subtracting angles is important.) In Figure 1.21b, the distance squared is

$$d^2 = (\text{change in } x)^2 + (\text{change in } y)^2$$

$$= (\cos s - \cos t)^2 + (\sin s - \sin t)^2. \tag{2}$$

Figure 1.21c shows the same circle and triangle (but rotated). The same distance squared is

$$d^2 = (\cos(s - t) - 1)^2 + (\sin(s - t))^2. \tag{3}$$

Now multiply out the squares in equations (2) and (3). Whenever $(\text{cosine})^2 + (\text{sine})^2$ appears, replace it by 1. The distances are the same, so (2) = (3):

$$(2) = 1 + 1 - 2 \cos s \cos t - 2 \sin s \sin t$$

$$(3) = 1 + 1 - 2 \cos(s - t).$$

After canceling $1 + 1$ and then -2, we have the "*addition formula*" for $\cos(s - t)$:

$$\text{The cosine of } s - t \text{ equals } \cos s \cos t + \sin s \sin t. \qquad (4)$$

$$\text{The cosine of } s + t \text{ equals } \cos s \cos t - \sin s \sin t. \qquad (5)$$

The easiest is $t = 0$. Then $\cos t = 1$ and $\sin t = 0$. The equations reduce to $\cos s = \cos s$.

To go from (4) to (5) in all cases, replace t by $-t$. No change in $\cos t$, but a "minus" appears with the sine. In the special case $s = t$, we have $\cos(t + t) = (\cos t)(\cos t) - (\sin t)(\sin t)$. This is a much-used formula for $\cos 2t$:

$$\textbf{\textit{Double angle}: } \cos 2t = \cos^2 t - \sin^2 t = 2\cos^2 t - 1 = 1 - 2\sin^2 t. \qquad (6)$$

I am constantly using $\cos^2 t + \sin^2 t = 1$, to switch between sines and cosines.

We also need addition formulas and double-angle formulas for the *sine* of $s - t$ and $s + t$ and $2t$. For that we connect sine to cosine, rather than $(\text{sine})^2$ to $(\text{cosine})^2$. The connection goes back to the ratio y/r in our original triangle. This is the sine of the angle θ and also the cosine of the *complementary angle* $\pi/2 - \theta$:

$$\sin \theta = \cos(\pi/2 - \theta) \qquad \text{and} \qquad \cos \theta = \sin(\pi/2 - \theta). \qquad (7)$$

The complementary angle is $\pi/2 - \theta$ because the two angles add to $\pi/2$ (a right angle). By making this connection in Problem 19, formulas (4–5–6) move from cosines to sines:

$$\sin(s - t) = \sin s \cos t - \cos s \sin t \qquad (8)$$

$$\sin(s + t) = \sin s \cos t + \cos s \sin t \qquad (9)$$

$$\sin 2t = \sin(t + t) = 2 \sin t \cos t \qquad (10)$$

I want to stop with these ten formulas, even if more are possible. Trigonometry is full of identities that connect its six functions—basically because all those functions come from a single right triangle. The x, y, r ratios and the equation $x^2 + y^2 = r^2$ can be rewritten in many ways. But you have now seen the formulas that are needed by calculus.† They give derivatives in Chapter 2 and integrals in Chapter 5. And it is typical of our subject to add something of its own—a limit in which an angle approaches zero. *The essence of calculus is in that limit.*

Review of the ten formulas   Figure 1.22 shows $d^2 = (0 - \tfrac{1}{2})^2 + (1 - \sqrt{3}/2)^2$.

$$\cos \frac{\pi}{6} = \cos \frac{\pi}{2} \cos \frac{\pi}{3} + \sin \frac{\pi}{2} \sin \frac{\pi}{3} \quad (s-t) \qquad \sin \frac{\pi}{6} = \sin \frac{\pi}{2} \cos \frac{\pi}{3} - \cos \frac{\pi}{2} \sin \frac{\pi}{3}$$

$$\cos \frac{5\pi}{6} = \cos \frac{\pi}{2} \cos \frac{\pi}{3} - \sin \frac{\pi}{2} \sin \frac{\pi}{3} \quad (s+t) \qquad \sin \frac{5\pi}{6} = \sin \frac{\pi}{2} \cos \frac{\pi}{3} + \cos \frac{\pi}{2} \sin \frac{\pi}{3}$$

$$\cos 2\frac{\pi}{3} = \cos^2 \frac{\pi}{3} - \sin^2 \frac{\pi}{3} \qquad (2t) \qquad \sin 2\frac{\pi}{3} = 2 \sin \frac{\pi}{3} \cos \frac{\pi}{3}$$

$$\cos \frac{\pi}{6} = \sin \frac{\pi}{3} = \sqrt{3}/2 \qquad \left(\frac{\pi}{2} - \theta\right) \qquad \sin \frac{\pi}{6} = \cos \frac{\pi}{3} = 1/2$$

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†Calculus turns (6) around to $\cos^2 t = \tfrac{1}{2}(1 + \cos 2t)$ and $\sin^2 t = \tfrac{1}{2}(1 - \cos 2t)$.
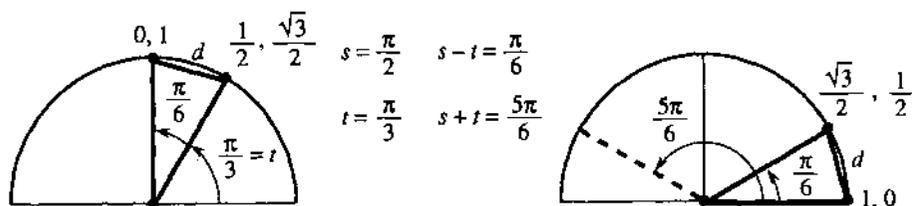
Fig. 1.22

1.5 EXERCISES

Read-through questions

Starting with a __a__ triangle, the six basic functions are the __b__ of the sides. Two ratios (the cosine x/r and the __c__) are below 1. Two ratios (the secant r/x and the __d__) are above 1. Two ratios (the __e__ and the __f__) can take any value. The six functions are defined for all angles θ, by changing from a triangle to a __g__.

The angle θ is measured in __h__. A full circle is $\theta =$ __i__, when the distance around is $2\pi r$. The distance to angle θ is __j__. All six functions have period __k__. Going clockwise changes the sign of θ and __l__ and __m__. Since $\cos(-\theta) = \cos\theta$, the cosine is __n__.

Coming from $x^2 + y^2 = r^2$ are the three identities $\sin^2\theta + \cos^2\theta = 1$ and __o__ and __p__. (Divide by r^2 and __q__ and __r__.) The distance from $(2, 5)$ to $(3, 4)$ is $d =$ __s__. The distance from $(1, 0)$ to $(\cos(s-t), \sin(s-t))$ leads to the addition formula $\cos(s-t) =$ __t__. Changing the sign of t gives $\cos(s+t) =$ __u__. Choosing $s = t$ gives $\cos 2t =$ __v__ or __w__. Therefore $\frac{1}{2}(1 + \cos 2t) =$ __x__, a formula needed in calculus.

1 In a 60–60–60 triangle show why $\sin 30° = \frac{1}{2}$.

2 Convert π, 3π, $-\pi/4$ to degrees and $60°$, $90°$, $270°$ to radians. What angles between 0 and 2π correspond to $\theta = 480°$ and $\theta = -1°$?

3 Draw graphs of $\tan\theta$ and $\cot\theta$ from 0 to 2π. What is their (shortest) period?

4 Show that $\cos 2\theta$ and $\cos^2\theta$ have period π and draw them on the same graph.

5 At $\theta = 3\pi/2$ compute the six basic functions and check $\cos^2\theta + \sin^2\theta$, $\sec^2\theta - \tan^2\theta$, $\csc^2\theta - \cot^2\theta$.

6 Prepare a table showing the values of the six basic functions at $\theta = 0$, $\pi/4$, $\pi/3$, $\pi/2$, π.

7 The area of a circle is πr^2. What is the area of the sector that has angle θ? It is a fraction _____ of the whole area.

8 Find the distance from $(1, 0)$ to $(0, 1)$ along (a) a straight line (b) a quarter-circle (c) a semicircle centered at $(\frac{1}{2}, \frac{1}{2})$.

9 Find the distance d from $(1, 0)$ to $(\frac{1}{2}, \sqrt{3}/2)$ and show on a circle why $6d$ is less than 2π.

10 In Figure 1.22 compute d^2 and (with calculator) $12d$. Why is $12d$ close to and below 2π?

11 Decide whether these equations are true or false:

(a) $\dfrac{\sin\theta}{1 - \cos\theta} = \dfrac{1 + \cos\theta}{\sin\theta}$

(b) $\dfrac{\sec\theta + \csc\theta}{\tan\theta + \cot\theta} = \sin\theta + \cos\theta$

(c) $\cos\theta - \sec\theta = \sin\theta\tan\theta$

(d) $\sin(2\pi - \theta) = \sin\theta$

12 Simplify $\sin(\pi - \theta)$, $\cos(\pi - \theta)$, $\sin(\pi/2 + \theta)$, $\cos(\pi/2 + \theta)$.

13 From the formula for $\cos(2t + t)$ find $\cos 3t$ in terms of $\cos t$.

14 From the formula for $\sin(2t + t)$ find $\sin 3t$ in terms of $\sin t$.

15 By averaging $\cos(s - t)$ and $\cos(s + t)$ in (4–5) find a formula for $\cos s \cos t$. Find a similar formula for $\sin s \sin t$.

16 Show that $(\cos t + i \sin t)^2 = \cos 2t + i \sin 2t$, if $i^2 = -1$.

17 Draw $\cos\theta$ and $\sec\theta$ on the same graph. Find all points where $\cos\theta = \sec\theta$.

18 Find all angles s and t between 0 and 2π where $\sin(s + t) = \sin s + \sin t$.

19 Complementary angles have $\sin\theta = \cos(\pi/2 - \theta)$. Write $\sin(s + t)$ as $\cos(\pi/2 - s - t)$ and apply formula (4) with $\pi/2 - s$ instead of s. In this way derive the addition formula (9).

20 If formula (9) is true, how do you prove (8)?

21 Check the addition formulas (4–5) and (8–9) for $s = t = \pi/4$.

22 Use (5) and (9) to find a formula for $\tan(s + t)$.

In 23–28 find *every* θ that satisfies the equation.

23 $\sin \theta = -1$ **24** $\sec \theta = -2$

25 $\sin \theta = \cos \theta$ **26** $\sin \theta = \theta$

27 $\sec^2 \theta + \csc^2 \theta = 1$ **28** $\tan \theta = 0$

29 Rewrite $\cos \theta + \sin \theta$ as $\sqrt{2} \sin(\theta + \phi)$ by choosing the correct "phase angle" ϕ. (Make the equation correct at $\theta = 0$. Square both sides to check.)

30 Match $a \sin x + b \cos x$ with $A \sin(x + \phi)$. From equation (9) show that $a = A \cos \phi$ and $b = A \sin \phi$. Square and add to find $A = \underline{\qquad}$. Divide to find $\tan \phi = b/a$.

31 Draw the base of a triangle from the origin $O = (0, 0)$ to $P = (a, 0)$. The third corner is at $Q = (b \cos \theta, b \sin \theta)$. What are the side lengths OP and OQ? From the distance formula

(1) show that the side PQ has length

$$d^2 = a^2 + b^2 - 2ab \cos \theta \quad \text{(law of cosines)}.$$

32 Extend the same triangle to a parallelogram with its fourth corner at $R = (a + b \cos \theta, b \sin \theta)$. Find the length squared of the other diagonal OR.

Draw graphs for equations 33–36, and mark three points.

33 $y = \sin 2x$ **34** $y = 2 \sin \pi x$

35 $y = \frac{1}{2} \cos 2\pi x$ **36** $y = \sin x + \cos x$

37 Which of the six trigonometric functions are infinite at what angles?

38 Draw rough graphs or computer graphs of $t \sin t$ and $\sin 4t \sin t$ from 0 to 2π.

1.6 A Thousand Points of Light

The graphs on the back cover of the book show $y = \sin n$. This is very different from $y = \sin x$. The graph of $\sin x$ is one continuous curve. By the time it reaches $x = 10{,}000$, the curve has gone up and down $10{,}000/2\pi$ times. Those 1591 oscillations would be so crowded that you couldn't see anything. The graph of $\sin n$ has picked 10,000 points from the curve—and for some reason those points seem to lie on more than 40 separate sine curves.

The second graph shows the first 1000 points. They *don't* seem to lie on sine curves. Most people see hexagons. *But they are the same thousand points!* It is hard to believe that the graphs are the same, but I have learned what to do. *Tilt the second graph and look from the side at a narrow angle.* Now the first graph appears. You see "diamonds." The narrow angle compresses the x axis—back to the scale of the first graph.
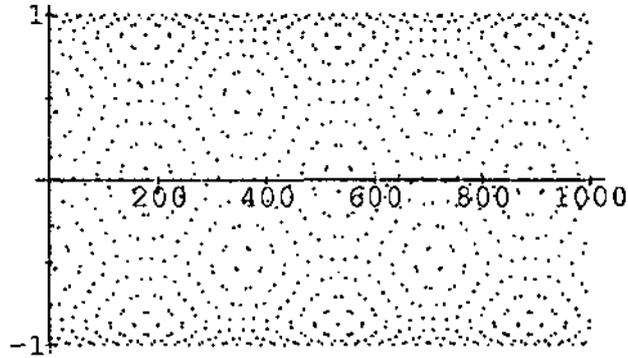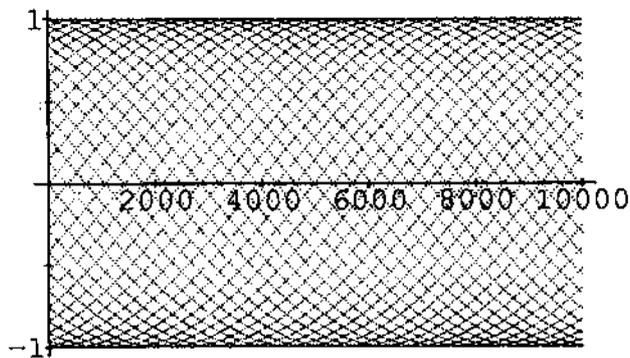
The effect of scale is something we don't think of. We understand it for maps. Computers can zoom in or zoom out—those are changes of scale. What our eyes see

depends on what is "close." We think we see sine curves in the 10,000 point graph, and they raise several questions:

1. Which points are near $(0, 0)$?
2. How many sine curves are there?
3. Where does the middle curve, going upward from $(0, 0)$, come back to zero?

A point near $(0, 0)$ really means that sin n is close to zero. That is certainly not true of sin 1 (1 is one radian!). In fact sin 1 is up the axis at .84, at the start of the seventh sine curve. Similarly sin 2 is .91 and sin 3 is .14. (The numbers 3 and .14 make us think of π. The sine of 3 equals the sine of $\pi - 3$. Then sin .14 is near .14.) Similarly sin 4, sin 5, ..., sin 21 are not especially close to zero.

The first point to come close is sin 22. This is because 22/7 is near π. Then 22 is close to 7π, whose sine is zero:

$$\sin 22 = \sin(7\pi - 22) \approx \sin(-.01) \approx -.01.$$

That is the first point to the right of $(0, 0)$ and slightly below. You can see it on graph 1, and more clearly on graph 2. It begins a curve downward.

The next point to come close is sin 44. This is because 44 is just past 14π.

$$44 \approx 14\pi + .02 \qquad \text{so} \qquad \sin 44 \approx \sin .02 \approx .02.$$

This point (44, sin 44) *starts the middle sine curve*. Next is (88, sin 88).

Now we know something. *There are 44 curves*. They begin near the heights sin 0, sin 1, ..., sin 43. Of these 44 curves, 22 start upward and 22 start downward. I was confused at first, because I could only find 42 curves. The reason is that sin 11 equals -0.99999 and sin 33 equals .9999. Those are so close to the bottom and top that you can't see their curves. The sine of 11 is near -1 because sin 22 is near zero. It is almost impossible to follow a single curve past the top—coming back down it is not the curve you think it is.

The points on the middle curve are at $n = 0$ and 44 and 88 and every number $44N$. Where does that curve come back to zero? In other words, when does $44N$ come *very close* to a multiple of π? We know that 44 is $14\pi + .02$. More exactly 44 is $14\pi + .0177$. So we multiply .0177 until we reach π:

$$\text{if} \quad N = \pi/.0177 \quad \text{then} \quad 44N = (14\pi + .0177)N = 14\pi N + \pi.$$

This gives $N = 177.5$. At that point $44N = 7810$. *This is half the period of the sine curve*. The sine of 7810 is very near zero.

If you follow the middle sine curve, you will see it come back to zero above 7810. The actual points on that curve have $n = 44 \cdot 177$ and $n = 44 \cdot 178$, with sines just above and below zero. Halfway between is $n = 7810$. *The equation for the middle sine curve is* $y = \sin(\pi x/7810)$. Its period is 15,620—beyond our graph.

Question The fourth point on that middle curve looks the same as the fourth point coming down from sin 3. What is this "double point?"
Answer 4 times 44 is 176. On the curve going up, the point is (176, sin 176). On the curve coming down it is (179, sin 179). *The sines of 176 and 179 differ only by* .00003.

The second graph spreads out this double point. Look above 176 and 179, at the center of a hexagon. You can follow the sine curve all the way across graph 2.

Only a little question remains. Why does graph 2 have hexagons? *I don't know.* The problem is with your eyes. To understand the hexagons, Doug Hardin plotted points on straight lines as well as sine curves. Graph 3 shows $y = $ fractional part of $n/2\pi$. Then he made a second copy, turned it over, and placed it on top. That produced graph 4—with hexagons. Graphs 3 and 4 are on the next page.
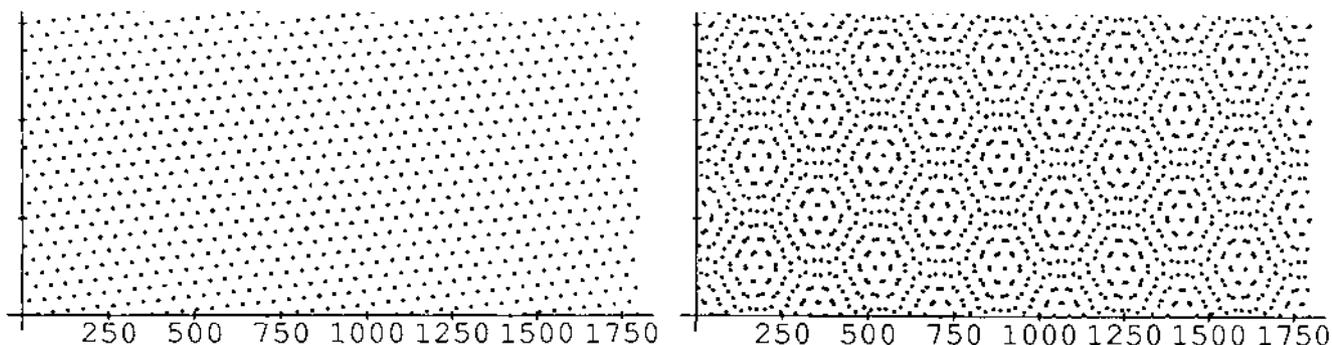
This is called a *Moiré pattern*. If you can get a transparent copy of graph 3, and turn it slowly over the original, you will see fantastic hexagons. They come from interference between periodic patterns—in our case 44/7 and 25/4 and 19/3 are near 2π. This interference is an enemy of printers, when color screens don't line up. It can cause vertical lines on a TV. Also in making cloth, operators get dizzy from seeing Moiré patterns move. There are good applications in engineering and optics —but we have to get back to calculus.

1.7 Computing in Calculus

Software is available for calculus courses—a lot of it. The packages keep getting better. Which program to use (if any) depends on cost and convenience and purpose. *How* to use it is a much harder question. These pages identify some of the goals, and also particular packages and calculators. Then we make a beginning (this is still Chapter 1) on the connection of computing to calculus.

The discussion will be informal. It makes no sense to copy the manual. Our aim is to support, with examples and information, the effort to use computing to help learning.

For calculus, *the greatest advantage of the computer is to offer graphics*. You see the function, not just the formula. As you watch, $f(x)$ reaches a maximum or a minimum or zero. A separate graph shows its derivative. Those statements are not 100% true, as everybody learns right away   as soon as a few functions are typed in. But the power to *see this subject* is enormous, because it is adjustable. If we don't like the picture we change to a new viewing window.

This is computer-based graphics. It combines *numerical* computation with *graphical* computation. You get pictures as well as numbers—a powerful combination. The computer offers the experience of actually working with a function. The domain and range are not just abstract ideas. *You choose them.* May I give a few examples.

EXAMPLE 1   Certainly x^3 equals 3^x when $x = 3$. *Do those graphs ever meet again?* At this point we don't know the full meaning of 3^x, except when x is a nice number. (Neither does the computer.) Checking at $x = 2$ and 4, the function x^3 is smaller both times: 2^3 is below 3^2 and $4^3 = 64$ is below $3^4 = 81$. If x^3 is always less than 3^x we ought to know—these are among the basic functions of mathematics.

The computer will answer numerically or graphically. At our command, it solves $x^3 = 3^x$. At another command, it plots both functions—this shows more. The screen proves a point of logic (or mathematics) that escaped us. If the graphs cross once, they must cross again—because 3^x is higher at 2 and 4. A crossing point near 2.5 is seen by zooming in. I am less interested in the exact number than its position—it comes before $x = 3$ rather than after.

A few conclusions from such a basic example:

1. A supercomputer is not necessary.
2. High-level programming is not necessary.
3. We can do mathematics without completely understanding it.

The third point doesn't sound so good. Write it differently: *We can learn mathematics while doing it*. The hardest part of teaching calculus is to turn it from a spectator sport into a workout. The computer makes that possible.

EXAMPLE 2   (mental computer) Compare x^2 with 2^x. The functions meet at $x = 2$. Where do they meet again? Is it before or after 2?

That is mental computing because the answer happens to be a whole number (4). Now we are on a different track. Does an accident like $2^4 = 4^2$ ever happen again? Can the machine tell us about integers? Perhaps it can plot the solutions of $x^b = b^x$. I asked *Mathematica* for a formula, hoping to discover x as a function of b—but the program just gave back the equation. For once the machine typed HELP instead of the user.

Well, mathematics is not helpless. I am proud of calculus. There is a new exercise at the end of Section 6.4, to show that we never see whole numbers again.

EXAMPLE 3   Find the number b for which $x^b = b^x$ has only *one* solution (at $x = b$).

When b is 3, the second solution is below 3. When b is 2, the second solution (4) is above 2. If we move b from 2 to 3, there must be a special "double point"—where the graphs barely touch but don't cross. For that particular b—and only for that one value—the curve x^b never goes above b^x.

This special point b can be found with computer-based graphics. In many ways it is the "*center point of calculus*." Since the curves touch but don't cross, they are tangent. They have the same slope at the double point. Calculus was created to work with slopes, and we already know the slope of x^2. Soon comes x^b. Eventually we discover the slope of b^x, and identify the most important number in calculus.

The point is that this number can be discovered first by experiment.

EXAMPLE 4   Graph $y(x) = e^x - x^e$. Locate its minimum.

The next example was proposed by Don Small. Solve $x^4 - 11x^3 + 5x - 2 = 0$. The first tool is algebra—try to factor the polynomial. That succeeds for quadratics, and then gets extremely hard. Even if the computer can do algebra better than we can, factoring is seldom the way to go. In reality we have two good choices:

1. (*Mathematics*) Use the derivative. Solve by Newton's method.
2. (*Graphics*) Plot the function and zoom in.

Both will be done by the computer. Both have potential problems! Newton's method is fast, but that means it can fail fast. (It is usually terrific.) Plotting the graph is also fast—but solutions can be outside the viewing window. This particular function is

zero only once, in the standard window from -10 to 10. The graph seems to be leaving zero, but mathematics again predicts a second crossing point. So we zoom out before we zoom in.

The use of the zoom is the best part of graphing. Not only do we *choose* the domain and range, we *change* them. The viewing window is controlled by four numbers. They can be the limits $A \leqslant x \leqslant B$ and $C \leqslant y \leqslant D$. They can be the coordinates of two opposite corners: (A, C) and (B, D). They can be the center position (a, b) and the scale factors c and d. Clicking on opposite corners of the zoom box is the fastest way, unless the center is unchanged and we only need to give scale factors. (Even faster: Use the default factors.) Section 3.4 discusses the *centering transform* and *zoom transform*—a change of picture on the screen and a change of variable within the function.

EXAMPLE 5 Find all real solutions to $x^4 - 11x^3 + 5x - 2 = 0$.

EXAMPLE 6 Zoom out and in on the graphs of $y = \cos 40x$ and $y = x \sin(1/x)$. Describe what you see.

EXAMPLE 7 What does $y = (\tan x - \sin x)/x^3$ become at $x = 0$? For small x the machine eventually can't separate $\tan x$ from $\sin x$. It may give $y = 0$. Can you get close enough to see the limit of y?

For these examples, and for most computer exercises in this book, a *menu-driven* system is entirely adequate. There is a list of commands to choose from. The user provides a formula for $y(x)$, and many functions are built in. A calculus supplement can be very useful—*MicroCalc* or *True* BASIC or *Exploring Calculus* or *MPP* (in the public domain). Specific to graphics are *Surface Plotter* and *Master Grapher* and *Gyrographics* (animated). The best software for linear algebra is *MATLAB*.

Powerful packages are increasing in convenience and decreasing in cost. They are capable of *symbolic* computation—which opens up a third avenue of computing in calculus.

SYMBOLIC COMPUTATION

In symbolic computation, answers can be *formulas* as well as numbers and graphs. The derivative of $y = x^2$ is seen as "$2x$." The derivative of $\sin t$ is "$\cos t$." The slope of b^x is known to the program. The computer does more than substitute numbers into formulas—it operates directly on the formulas. We need to think where this fits with learning calculus.

In a way, symbolic computing is close to what we ourselves do. Maybe too close—there is some danger that symbolic manipulation is *all* we do. With a higher-level language and enough power, a computer can print the derivative of $\sin(x^2)$. So why learn the chain rule? Because mathematics goes deeper than "algebra with formulas." We deal with *ideas*.

I want to say clearly: Mathematics is not formulas, or computations, or even proofs, but ideas. The symbols and pictures are the language. The book and the professor and the computer can join in teaching it. The computer should be non-threatening (like this book and your professor)—you can work at your own pace. Your part is to learn by doing.

EXAMPLE 8 A computer algebra system quickly finds 100 factorial. This is $100! = (100)(99)(98) \dots (1)$. The number has 158 digits (not written out here). The last 24

digits are zeros. For $10! = 3628800$ there are seven digits and two zeros. Between 10 and 100, and beyond, are simple questions that need ideas:

1. How many digits (approximately) are in the number $N!$?
2. How many zeros (exactly) are at the end of $N!$?

For Question 1, the computer shows more than N digits when $N = 100$. It will never show more than N^2 digits, because none of the N terms can have more than N digits. A much tighter bound would be $2N$, but is it true? *Does $N!$ always have fewer than $2N$ digits?*

For Question 2, the zeros in $10!$ can be explained. One comes from 10, the other from 5 times 2. (10 is also 5 times 2.) Can you explain the 24 zeros in $100!$? An idea from the card game blackjack applies here too: *Count the fives.*

Hard question: How many zeros at the end of $200!$?

The outstanding package for full-scale symbolic computation is *Mathematica.* It was used to draw graphs for this book, including $y = \sin n$ on the back cover. *The complete command was* ListPlot[Table [Sin[n], {n, 10000}]]. This system has rewards and also drawbacks, including the price. Its original purpose, like *MathCAD* and *MACSYMA* and *REDUCE*, was not to teach calculus—but it can. The computer algebra system *MAPLE* is good.

As I write in 1990, DERIVE is becoming well established for the PC. For the Macintosh, *Calculus T/L* is a "sleeper" that deserves to be widely known. It builds on *MAPLE* and is much more accessible for calculus. An important alternative is *Theorist.* These are menu-driven (therefore easier at the start) and not expensive.

I strongly recommend that students share terminals and work together. Two at a terminal and 3–5 in a working group seems to be optimal. Mathematics can be learned by *talking* and *writing*—it is a human activity. Our goal is not to test but to teach and learn.

Writing in Calculus May I emphasize the importance of writing? We totally miss it, when the answer is just a number. A one-page report is harder on instructors as well as students—but much more valuable. A word processor keeps it neat. You can't write sentences without being forced to organize ideas—and part of yourself goes into it.

I will propose a writing exercise with options. If you have computer-based graphing, follow through on Examples 1–4 above and report. Without a computer, pick a paragraph from this book that should be clearer and *make it clearer.* Rewrite it with examples. Identify the key idea at the start, explain it, and come back to express it differently at the end. Ideas are like surfaces—they can be seen many ways.

Every reader will understand that in software there is no last word. New packages keep coming (*Analyzer* and *EPIC* among them). The biggest challenges at this moment are three-dimensional graphics and calculus workbooks. In $3D$, the problem is the position of the eye—since the screen is only $2D$. In workbooks, the problem is to get past symbol manipulation and reach ideas. Every teacher, including this one, knows how hard that is and hopes to help.

GRAPHING CALCULATORS

The most valuable feature for calculus—*computer-based graphing*—is available on hand calculators. With trace and zoom their graphs are quite readable. By creating the graphs you subconsciously learn about functions. These are genuinely *personal* computers, and the following pages aim to support and encourage their use.

Programs for a hand-held machine tend to be simple and short. We don't count the zeros in 100 factorial (probably we could). A calculator finds crossing points and maximum points to good accuracy. Most of all it allows you to explore calculus by yourself. You set the viewing window and define the function. Then you see it.

There is a choice of calculators—*which one to buy?* For this book there was also a choice—*which one to describe?* To provide you with listings for useful programs, we had to choose. Fortunately the logic is so clear that you can translate the instructions into any language—for a computer as well as a calculator. The programs given here are the "greatest common denominator" of computing in calculus.

The range of choices starts with the Casio *fx* 7000G—the first and simplest, with very limited memory but a good screen. The Casio 7500, 8000, and 8500 have increasing memory and extra features. The Sharp *EL*-5200 (or 9000 in Canada and Europe) is comparable to the Casio 8000. These machines have *algebraic entry*—the normal order as in $y = x + 3$. They are inexpensive and good. More expensive and much more powerful are the Hewlett-Packard calculators—the *HP-28S* and *HP-48SX*. They have large memories and extensive menus (and symbolic algebra). They use *reverse Polish notation*—numbers first in the stack, then commands. They require extra time and effort, and other books do justice to their amazing capabilities. It is estimated that those calculators could get 95 on a typical calculus exam.

While this book was being written, Texas Instruments produced a new graphing calculator: the *TI*-81. It is closer to the Casio and Sharp (emphasis on graphing, easy to learn, no symbolic algebra, moderate price). With earlier machines as a starting point, many improvements were added. There is some risk in a choice that is available only Δt before this textbook is published, and we hope that the experts we asked are right. Anyway, *our programs are for the TI*-81. It is impressive.

These few pages are no substitute for the manual that comes with a calculator. A valuable supplement is a guide directed especially at calculus—my absolute favorites are *Calculus Activities for Graphic Calculators* by Dennis Pence (PWS-Kent, 1990 for the Casio and Sharp and *HP-28S*, 1991 for the *TI*-81). A series of *Calculator Enhancements*, using *HP*'s, is being published by Harcourt Brace Jovanovich. What follows is an introduction to one part of a calculus laboratory. Later in the book, we supply *TI*-81 programs close to the mathematics and the exercises that they are prepared for.

A few words to start: To select from a menu, press the item number and E N T E R. Edit a command line using D E L(ete) and I N S(ert). Every line ends with E N T E R. For calculus select radians on the M O D E screen. For powers use ^ . For special powers choose x^2, x^{-1}, $\sqrt{x}$. Multiplication has priority, so $(-)3 + 2 \times 2$ produces 1. Use keys for S I N, I F, I S, ... When you press letters, I multiplies S.

If a program says $3 \rightarrow C$, type 3 S T O C E N T E R. Storage locations are A to Z or Greek θ.

Functions A graphing calculator helps you (forces you?) to understand the concept of a function. It also helps you to understand specific functions—especially when changing the viewing window.

To evaluate $y = x^2 - 2x$ *just once*, use the home screen. To define $y(x)$ for *repeated use*, move to the function edit screen: Press M O D E, choose F u n c t i o n, and press Y =. Then type in the formula. *Important tip*: for X on the TI-81, the key X|T is faster than two steps A l p h a X. The Y = edit screen is the same place where the formula is needed for graphing.

Example $Y_1 = X^2 - 2X$ ENTER on the Y= screen. 4 STO X ENTER on the home screen. Y_1 ENTER on the Y-VARS screen. The screen shows 8, which is $Y(4)$. The formula remains when the calculator is off.

Graphing You specify the X range and Y range. (We should say X domain but we don't.) The screen is a grid of 96×64 little rectangles called "*pixels.*" The first column of pixels represents $X\min$ and the last column is $X\max$. Press RANGE to reset. With $X res = 1$ the function is evaluated 96 times as it is graphed. $X scl$ and $Y scl$ give the spaces between ticks on the axes.

The ZOOM menu is a fast way to set ranges. ZOOM Standard gives the default $-10 \leqslant x \leqslant 10$, $-10 \leqslant y \leqslant 10$. ZOOM Trig gives $-2\pi \leqslant x \leqslant 2\pi$, $-3 \leqslant y \leqslant 3$.

The keystroke GRAPH shows the graphing screen with the current functions.

Example Set the ranges $(-)2 \leqslant X \leqslant 3$ and $(-)150 \leqslant Y \leqslant 50$. Press Y= and store $Y_1 = X$ (in MATH)$^3 - 28X^2 + 15X + 36$ ENTER. Press GRAPH. You won't see much of the graph! Press RANGE and reset $(-)10 \leqslant X \leqslant 30, (-)4000 \leqslant Y \leqslant 2300$. Press GRAPH. See a cubic polynomial.

"Smart Graph" recalls the graph instantly without redrawing it, if no settings have changed. The DRAW menu is for points, lines, and shaded regions. This is perfect for our piecewise linear functions—just connect the breakpoints with lines. In Section 3.6 the lines show an iteration by its "cobweb."

Programming This book contains programs that you can type in once and save. We chose *Autoscaling, Newton's Method, Secant Method, Cobweb Iteration,* and *Numerical Integration.* You will create others—to do calculations or to add features that are not available as single keystrokes. The calculator is like a computer, with a fairly small set of instructions. *One difference*: Memory is too precious to store comments with the code. You have to see the logic by rereading the program.

To enter the world of programming, press PRGM. Each PRGM submenu lists all programs by *name*—a digit, a letter, or θ (37 names). The program *title* has up to eight characters. Select the EDIT submenu and press G for the edit screen. Type the title GRAPHS and press ENTER. Practice on this one:

```
:"X²+X" STO (Y-VARS) Y₁ ENTER
:"X-1" STO (Y-VARS) Y₂ ENTER
:(PRGM)(I/O) DispGraph
```

The menus to call are in parentheses. Leave the edit screen with QUIT (not CLEAR—that erases the line with the cursor). Set the default window by ZOOM Standard.

To execute, press PRGM (EXEC) G ENTER. The program draws the graphs. It leaves Y_1 and Y_2 on the Y= screen. To erase the program from the home screen, press (PRGM)(ERASE)G. Practice again by creating Prgm2:FUNC. Type $:\sqrt{}$ X STO Y and :(PRGM)(I/O)Disp Y. Move to the home screen, store X by 4 STO X ENTER, and execute by (PRGM)(EXEC)2 ENTER. Also try $X = -1$. When it fails to imagine i, select 1:Goto Error.

Piecewise functions and *Input* (to a running program). The definition of a piecewise function includes the *domain of each piece.* Logical tests like "IF $X \geqslant 7$" determine which domain the input value X falls into. An IF statement only affects the following line—which is executed when TEST = 1 (meaning *true*) and skipped when TEST = 0 (meaning *false*). IF commands are in the PRGM (CTL) submenu; TEST calls the menu of inequalities.

An input value $X = 4$ need not be stored in advance. Program P stops while running to request input. Execute with P ENTER after selecting the PRGM (EXEC) menu. Answer ? with 4 and ENTER. After completion, rerun by pressing ENTER again. The function is $y = 14 - x$ if $x < 7$, $y = x$ *if* $x \geqslant 7$.

```
PrgmP: PIECES
:Disp "X="             PGRM (I/O) Ask for input
:Input X               PGRM (I/O) Screen ? ENTER X
:14-X → Y              First formula for all X
:If 7 ≤ X              PRGM (CTL) TEST
:X → Y                 Overwrite if TEST = 1
:Disp Y                Display Y(X)
```

Overwriting is faster than checking both ends $A \leqslant X \leqslant B$ for each piece. Even faster: a whole formula $(14 - X)(X < 7) + (X)(7 \leqslant X)$ can go on a single line using 1 and 0 from the tests. Compute-store-display $Y(X)$ as above, or define Y1 on the edit screen.

Exercise   Define a third piece $Y = 8 + X$ if $X < 3$. Rewrite P using Y1 = . A product of tests $(3 \leqslant X)$ $(X < 7)$ evaluates to 1 if *all true* and to 0 if *any false*.

TRACE *and* ZOOM   The best feature is graphing. But a whole graph can be like a whole book—too much at once. You want to focus on one part. A computer or calculator will trace along the graph, stop at a point, and zoom in.

There is also ZOOM OUT, to widen the ranges and see more. Our eyes work the same way—they put together information on different scales. Looking around the room uses an amazingly large part of the human brain. With a big enough computer we can try to imitate the eyes—this is a key problem in artificial intelligence. With a small computer and a zoom feature, we can *use* our eyes to understand functions.

Press TRACE to locate a point on the graph. A blinking cursor appears. Move left or right—the cursor stays on the graph. Its coordinates appear at the bottom of the screen. When x changes by a pixel, the calculator evaluates $y(x)$. To solve $y(x) = 0$, read off x at the point when y is nearest to zero. To minimize or maximize $y(x)$, read off the smallest and largest y. In all these problems, zoom in for more accuracy.

To blow up a figure we can choose new ranges. The fast way is to use a ZOOM command. For a preset range, use ZOOM Standard or ZOOM Trig. To shrink or stretch by XFact or YFact (default values 4), use ZOOM In or ZOOM Out. Choose the center point and press ENTER. The new graph appears. Change those scaling factors with ZOOM Set Factors. Best of all, *create your own viewing window*. Press ZOOM Box.

To draw the box, move the cursor to one corner. Press ENTER and this point is a small square. The same keys move a second (blinking) square to the opposite corner—the box grows as you move. Press ENTER, and the box is the new viewing window. The graphs show the same function with a change of scale. Section 3.4 will discuss the mathematics—here we concentrate on the graphics.

EXAMPLE 9   Place :Y1 =X sin (1/X) in the Y= edit screen. Press ZOOM Trig for a first graph. Set XFact = 1 and YFact = 2.5. Press ZOOM In with center at $(0, 0)$. To see a larger picture, use XFact = 10 and YFact = 1. Then Zoom Out again. As X gets large, the function $X \sin(1/X)$ approaches _____.

Now return to ZOOM Trig. Zoom In with the factors set to 4 (default). Zoom again by pressing ENTER. With the center and the factors fixed, this is faster than drawing a zoom box.

EXAMPLE 10 Repeat for the more erratic function $Y = \sin(1/X)$. After ZOOM Trig, create a box to see this function near $X = .01$. The Y range is now _____.

Scaling is crucial. For a new function it can be tedious. A formula for $y(x)$ does not easily reveal the range of y's, when $A \leqslant x \leqslant B$ is given. The following program is often more convenient than zooms. It samples the function $L = 19$ times across the x-range (every 5 pixels). The inputs *Xmin, Xmax, Y_1* are previously stored on other screens. After sampling, the program sets the y-range from $C = Ymin$ to $D = Ymax$ and draws the graph.

Notice the *loop* with counter K. The loop ends with the command I S > (K , L), which increases K by 1 and skips a line if the new K exceeds L. Otherwise the command Goto 1 restarts the loop. The screen shows the short form on the left.

Example: $Y_1 = x^3 + 10x^2 - 7x + 42$ with range Xmin $= -12$ and Xmax $= 10$. Set tick spacing Xscl $= 4$ and Yscl $= 250$. Execute with PRGM (EXEC) A ENTER. For this program we also list menu locations and comments.

PrgmA : AUTOSCL	Menu (Submenu) Comment
: ALL-Off	Y-VARS (OFF) Turn off functions
: Xmin → A	VARS (RNG) Store Xmin using STO
: 19 → L	Store number of evaluations (19)
: (Xmax-A)/L → H	Spacing between evaluations
: A → X	Start at $x = A$
: Y_1 → C	Y-VARS (Y) Evaluate the function
: C → D	Start C and D with this value
: 1 → K	Initialize counter $K = 1$
: Lbl 1	PRGM (CTL) Mark loop start
: A+KH → X	Calculate next x
: Y_1 → Y	Evaluate function at x
: IF Y<C	PGRM (CTL) New minimum?
: Y → C	Update C
: IF D<Y	PRGM (CTL) New maximum?
: Y → D	Update D
: IS> (K,L)	PRGM (CTL) Add 1 to K, skip Goto if $>L$
: Goto 1	PRGM (CTL) Loop return to Lbl 1
: Y_1-On	Y-VARS (ON) Turn on Y_1
: C → Ymin	STO VARS (RNG) Set Ymin $= C$
: D → Ymax	STO VARS (RNG) Set Ymax $= D$
: DispGraph	PRGM (I/O) Generate graph

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Resource: Calculus Online Textbook
Gilbert Strang

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