Chapter 3: Duality Toolbox

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Lecture 19

3.1.5: MASS-DIMENSION RELATION

We now consider the following questions:

1. How the conformal dimension of an operator is mapped to the bulk?
2. How to interpret modes of bulk field in the boundary theory?

On gravity side, the action generally has the form:

$$S = \frac{1}{2\kappa^2}\int d^{d+1}x\sqrt{-g}\left(\mathcal{R} - 2\Lambda + \mathcal{L}_{matter}\right) \qquad 2\kappa^2 = 16\pi G_{d+1} \tag{1}$$

where the matter Lagrangian is

$$\mathcal{L}_{matter} = -\frac{1}{2}(\partial\Phi)^2 - \frac{1}{2}m^2\Phi^2 - V(\Phi) + \cdots \tag{2}$$

where $V(\Phi)$ includes non-linear terms.

Consider small fluctuations of Φ and g_{MN}, then it is convenient to normalize them canonically:

$$\Phi \to \kappa\Phi \qquad g_{MN} = g^{(0)}_{MN} + \kappa h_{MN} \tag{3}$$

Remember in the matching of parameters, $\kappa \sim G^{1/2}_{d+1} \sim N^{-1}$ in the unit of curvature is small in large N limit. Thus for Φ, $h_{MN} \sim O(1)$, nonlinear terms are $O(\kappa)$ or higher and can be neglected. At leading order, (2) becomes a quadratic action in AdS, *i.e.* free theory.

We use a massive scalar field for illustration. The bulk quadratic action for Φ is

$$S = -\frac{1}{2}\int d^{d+1}x\sqrt{-g}(g^{MN}\partial_M\Phi\partial_N\Phi + m^2\Phi) + O(\kappa) \tag{4}$$

where

$$ds^2 = \frac{R^2}{z^2}(-dt^2 + d\vec{x}^2 + dz^2) \qquad x^M = (z, x^\mu) \tag{5}$$

Then equation of motion of Φ is

$$\frac{1}{\sqrt{-g}}\partial_M(\sqrt{-g}g^{MN}\partial_N\Phi) - m^2\Phi = 0 \tag{6}$$

Given translation symmetries in x^μ directions, we make the ansatz for the solution:

$$\Phi(z, x^\mu) = \int \frac{d^dk}{(2\pi)^4}e^{ik\cdot x}\Phi(z;k) \qquad (k\cdot x = \eta_{\mu\nu}k^\mu x^\nu) \tag{7}$$

Substitute this ansatz into (6), we get

$$z^{d+1}\partial_z(z^{1-d}\partial_z\Phi) - k^2z^2\Phi - m^2R^2\Phi = 0 \tag{8}$$

where $k^\mu = (\omega, \vec{k})$ and $k^2 = -\omega^2 + \vec{k}^2$. Considering asymptotic behavior of Φ as $z \to 0$, one can drop the term involving k^2 that is $O(z^2)$ in (8), then get

$$z^2\partial_z^2\Phi + (1-d)z\partial_z\Phi - m^2R^2\Phi = 0 \tag{9}$$

This is a homogeneous equation, which can be solved by power function. Let $\Phi \sim z^\Delta$, (9) reduced to

$$\Delta(\Delta - 1) + (1-d)\Delta - m^2R^2 = 0 \tag{10}$$

which implies two solutions for Δ

$$\Delta = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 R^2} \tag{11}$$

Introducing the convention

$$\Delta \equiv \frac{d}{2} + v \qquad v \equiv \sqrt{\frac{d^2}{4} + m^2 R^2} \qquad \Delta_- \equiv \frac{d}{2} - v = d - \Delta \tag{12}$$

we can write down the asymptotic behavior of Φ:

$$\Phi(k, z) = A(k) z^{d-\Delta} + B(k) z^{\Delta} \qquad z \to 0 \tag{13}$$

$$\Phi(x, z) = A(x) z^{d-\Delta} + B(x) z^{\Delta} \qquad z \to 0 \tag{14}$$

Remarks

1. The exponents are real provided that

$$m^2 R^2 \geq -\frac{d^2}{4} \tag{15}$$

One can show that a theory is stable if and only if (15) is satisfied. When this condition is violated, there exist modes exponentially growing in time (see Nabil's notes) and it is called Breitenlohner-Freedman (BF) bound. Compare with the equation of motion in Minkowski spacetime:

$$\partial^2 \Phi - m^2 \Phi = 0 \implies \omega^2 = \vec{k}^2 + m^2 \tag{16}$$

and if $m^2 < 0$, continuity of $\vec{k}$ makes some $|\vec{k}|^2 < |m^2|$, thus $\omega^2 < 0$ and the solution will be

$$\Phi \sim e^{|\omega| t} + e^{-|\omega| t} \tag{17}$$

This is a tachyonic solution which will cause instability of the system and therefore must be excluded in the spectrum. This can be seen by noticing that in a scalar QFT in flat spacetime with general potential $V(\Phi)$, the vanishing first derivative of potential determines the vacuum expectation Φ_c, *i.e.* $V'(\Phi_c) = 0$, and the second derivative is the mass square of the fluctuation, *i.e.* $V''(\Phi_c) = m^2 < 0$, which implies the vacuum on the peak of the potential is unstable. However, in AdS, due to spacetime curvature, constant modes (like those $\vec{k} = 0$ in flat spacetime) are not allowed. A field is forced to have some kinetic energy, which can compensate some negative m^2.

2. AdS has a boundary and light rays reach the boundary in a finite time, *i.e.* energy may be exchanged at the boundary. We need to impose appropriate boundary conditions to have a well-defined notion of energy, which is also explained in detail in Nabil's notes.
Before we go any further, let me review some facts of canonical quantization of Φ in curved spacetime. If we expand Φ in term of a complete set of normalizable modes $\{u_i\}$, we must require them to satisfy the appropriate boundary condition. The normalizability is defined by finiteness of the following Klein-Gorden (KG) inner product:

$$(\Phi_1, \Phi_2) = -i \int_\Sigma dz d\vec{x} \sqrt{-\gamma} n^\mu (\Phi_1^* \partial_\mu \Phi_2 - \Phi_2 \partial_\mu \Phi_1^*) \tag{18}$$

where Σ is a spacelike hypersurface with induced metric γ_{ij} and unit normal vector n^μ. This inner product is independent of choice of Σ if flux leaking through boundary is zero:

$$\sqrt{\gamma_z} n_z^\mu (\Phi_1^* \partial_\mu \Phi_2 - \Phi_2 \partial_\mu \Phi_1^*) \to 0 \qquad (z \to 0,\ n_z^\mu \partial_\mu = z\partial_z,\ \gamma_z \text{ is induced boundary metric}) \tag{19}$$

because in this case the difference of two choices of Σ will be a bulk integral of the divergent of current $(\Phi_1^* \partial_\mu \Phi_2 - \Phi_2 \partial_\mu \Phi_1^*)$ by Gauss Law:

$$\delta(\Phi_1, \Phi_2) = -i \int_{\delta\Sigma} d^{d+1}x \sqrt{-g} \nabla^\mu (\Phi_1^* \nabla_\mu \Phi_2 - \Phi_2 \nabla_\mu \Phi_1^*) \tag{20}$$

where $\delta\Sigma$ means the bulk region surrounded by two Σs. Since

$$\nabla^\mu (\Phi_1^* \nabla_\mu \Phi_2 - \Phi_2 \nabla_\mu \Phi_1^*) = \Phi_1^* \nabla^2 \Phi_2 - \Phi_2 \nabla^2 \Phi_1^* = \Phi_1^* m^2 \Phi_2 - \Phi_2 m^2 \Phi_1^* = 0 \tag{21}$$

we see the KG inner product is independent of choice of Σ. The boundary condition (19) is also important for a well-defined notion of energy (Nabil's notes) because it implies the energy conservation in AdS (no leaking through boundary). In the quantization of QFT in curved spacetime, generally we can expand a scalar field Φ as

$$\Phi(x,z) = \sum_n \left(\phi_n(x,z)a_n + \phi_n^*(x,z)a_n^\dagger\right) \tag{22}$$

where n represents all quantum numbers, ϕ_n is the normalizable modes, which is normalized as $(\phi_n, \phi_m)_{KG} = \delta_{nm}$, and $a_n(a_n^\dagger)$ is the corresponding annihilation (creation) operator. As explained in first chapter, the definition of $a_n(a_n^\dagger)$ is not uniquely defined. Bogoliubov transformations of $a_n(a_n^\dagger)$ will give new sets of annihilation (creation) operators that correspond to different vacua.
Simple calculation shows that for real v, $\Delta \geq d/2$ and hence

$$z^\Delta \text{mode :always normalizable}$$

$$z^{d-\Delta}\text{mode:} \begin{cases} \text{normalizable} & 0 \leq v < 1 \\ \text{non-normalizable} & v \geq 1 \end{cases}$$

We can also check that only normalizable modes satisfy boundary condition (19), which suggests we can have well-defined energy in quantization and Hilbert space. Regarding different values of v, we can get different boundary conditions in (14) for normalizable modes for different quantization scheme:

$$\text{If } v \geq 1,\ A(x) = 0 \tag{23}$$

$$\text{If } 0 \leq v < 1,\ \begin{cases} A(x) = 0 & \text{standard quantization} \\ B(x) = 0 & \text{alternative quantization} \end{cases} \tag{24}$$

where in (24), it is also possible to impose mixed boundary conditions. In $0 \leq v < 1$ case, we will have two different quantization choices that will correspond to two interpretations for "normalizable". In order to get rid of this ambiguity, from now on, we will refer the word "normalizable" to a specific quantization (compatible with the boundary condition we choose for quantized mode) and refer the word "non-normalizable" to the other mode (no matter it can actually be normalized or not).

3. Normalizable modes are used to build up Hilbert space in the bulk via operators in (22). In AdS/CFT correspondence, we naturally expect bulk states should also have their boundary counterparts. Hence we must expect $\Phi(x,z) \overset{\sim}{\to} \mathcal{O}(x)$ for *normalizable* operator $\Phi(x,z)$. In other words, the Hilbert space in bulk is (expect to be) dual to that in boundary.

4. Non-normalizable modes are not part of Hilbert space. If present, they should be considered as defining background. This is perfectly consistent with earlier discussion. As seen in last lecture, $A(x)$ is the boundary "value" of the field (say in standard quantization). If $A(x) = \phi_0(x)$, we should add a term $\int d^dx\phi_0(x)\mathcal{O}(x)$ to boundary action, which is a modification of the boundary theory itself by a classical source $\phi_0(x)$, *i.e.* $\phi_0(x)$ is not quantized in boundary theory. More precisely, we have following correspondence for non-normalizable modes:

$$\int d^dx\phi_0(x)\mathcal{O}(x) \iff \phi_0(x) = \lim_{z\to 0} z^{\Delta-d}\Phi(z,x) \tag{25}$$

5. Relation (25) implies that Δ is the scaling dimension of $\mathcal{O}(x)$. In CFT, scaling dimension of $\mathcal{O}(x)$ is defined by:

$$x^\mu \to x'^\mu = \lambda x^\mu \qquad \mathcal{O}(x) \to \mathcal{O}'(x') = \lambda^{-\Delta}\mathcal{O}(x) \tag{26}$$

Bulk isometry

$$x^\mu \to x'^\mu = \lambda x^\mu,\ z \to z' = \lambda z \tag{27}$$

implies the scale transformation of (26) on boundary. Since bulk isometry is a part of diffeomorphism of bulk theory, and scaling transformation is a part of conformal symmetry of boundary theory, the AdS/CFT correspondence must still be valid, namely

$$\Phi(z,x) \iff \mathcal{O}(x) \to \Phi'(z',x') \iff \mathcal{O}'(x') \tag{28}$$

The fact that Φ is a scalar and boundary action should be conformal invariant implies

$$\Phi(z,x) = \Phi'(z',x') \qquad \int d^d x \phi_0(x)\mathcal{O}(x) = \int d^d x' \phi_0'(x')\mathcal{O}'(x') \tag{29}$$

On the other hand, (25) implies

$$\phi_0'(x') = \lim_{z\to 0} z'^{\Delta-d}\Phi(z',x') = \lambda^{\Delta-d}\phi_0(x) \tag{30}$$

Together with (29), we have

$$\int d^d x' \phi_0'(x')\mathcal{O}'(x') = \lambda^{\Delta}\int d^d x \phi_0(x)\mathcal{O}'(x') = \int d^d x \phi_0(x)\mathcal{O}(x) \implies \mathcal{O}'(x') = \lambda^{-\Delta}\mathcal{O}(x) \tag{31}$$

Hence, we find for a *i.e.* scalar field, in standard quantization,

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2R^2} \tag{32}$$

that implies the following three cases for boundary operators:
i). $m = 0$, $\Delta = d$, marginal operator;
ii). $m^2 < 0$, $\Delta < d$, relevant operator;
iii). $m^2 > 0$, $\Delta > d$, irrelevant operator.
Considering IR/UV connection, the bulk interpretation of above three operators is also clear: UV in boundary corresponds to $z \to 0$ in bulk, and since $\Phi(z,x) \to \phi_0(x)z^{d-\Delta}$, we see the asymptotic behavior in i) is constant (marginal), in ii) is vanishing (relevant, less and less important in UV) and in iii) is growing (irrelevant, more and more important in UV).

To summarize, consider a bulk scalar field $\Phi(x,z)$ of mass m, when $z \to 0$, Φ has two components

$$\Phi(z,x) = A(x)z^{d-\Delta} + B(x)z^{\Delta} + \cdots \tag{33}$$

where

$$\Delta = \frac{d}{2} + v \qquad v = \sqrt{\frac{d^2}{4} + m^2R^2} \tag{34}$$

We have following correspondence between bulk and boundary:

$$\begin{aligned} \Phi(x,z) &\iff \mathcal{O}(x) \\ \text{normalizable modes} &\iff \text{states} \\ \text{non-normalizable modes} &\iff \text{action} \end{aligned}$$

In standard quantization: ($A(x)$ is non-normalizable)

$$\begin{aligned} A(x) &\iff \int d^d x \phi_0(x)\mathcal{O}(x),\ \phi_0(x) = A(x) \\ B(x) &\iff \mathcal{O}(x)\ (B(x) \sim \langle \mathcal{O}(x)\rangle \text{to be shown later}) \\ m &\iff \Delta \end{aligned}$$

In alternative quantization: ($B(x)$ is non-normalizable, only for $0 < v < 1$)

$$\begin{aligned} B(x) &\iff \int d^d x \phi_0(x)\mathcal{O}(x),\ \phi_0(x) = A(x) \\ A(x) &\iff \mathcal{O}(x)\ (A(x) \sim \langle \mathcal{O}(x)\rangle \text{to be shown later}) \\ m &\iff d - \Delta \end{aligned}$$

In the last part of this lecture, I may give you some more examples to show the correspondence above is not restricted to scalar field. Let us consider Maxwell field A_M, which corresponds to J^μ. When $z \to 0$, the transverse component of the solution of equation of motion of A_M is

$$A_\mu = a_\mu + b_\mu z^{d-2} \tag{35}$$

where a_μ is non-normalizable mode. After similar scaling argument of $\int d^d x a_\mu J^\mu$, we can get $\Delta = d-1$ as expected for a conserved current. We can extend this discussion to massive vector field and get $\Delta = \frac{d}{2} + \sqrt{\frac{(d-2)^2}{4} + m^2R^2}$. For stress tensor $T_{\mu\nu}$, after setting the metric to be $ds^2 = f(z)dz^2 + g_{\mu\nu}dx^\mu dx^\nu$ where $g_{\mu\nu} \to \frac{R^2}{z^2}(\eta_{\mu\nu} + h_{\mu\nu})$, we can get $\Delta = d$ from scaling transformation in $\int d^d x h_{\mu\nu}T^{\mu\nu}$.

References

[1] An introduction to general relativity, spacetime and geometry. Sean M. Carroll.