

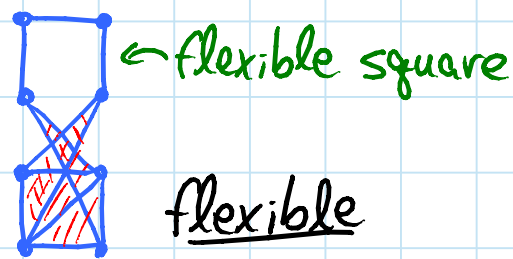
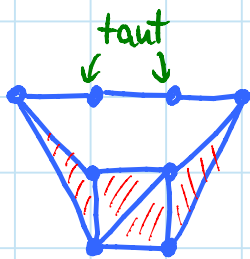
Review: (but without pinning)

Linkage = graph + lengths of edges $l: E \rightarrow \mathbb{R}^{\geq 0}$

Configuration of linkage = coordinates for vertices
 $C: V \rightarrow \mathbb{R}^d$ satisfying edge lengths

Rigidity: a linkage configuration is

- flexible if there's a nontrivial motion starting there
 - positive duration
 - not a rigid motion (translation + rotation)
- rigid otherwise (\emptyset D.O.F. / isolated point)
- depends on configuration, not just linkage:



- testing for given configuration is coNP-hard
 [Abbott, Barton, Demaine 2008]

Applications:

- architecture [e.g. Why Buildings Fall Down]
- biology (which parts of atom-bond structure move?)
- linkage folding [see L10] → [Michael Thorpe]

Generic rigidity: rigidity is usually a property of the graph, not the configuration or linkage

Realization of a graph

= coordinates for vertices $C: V \rightarrow \mathbb{R}^d$ (= pt. in \mathbb{R}^{dn})
= configuration of a linkage of the graph

Generic realization avoids any "degeneracies":

eg. {
- 3 points collinear
- 4 points concyclic, or inducing parallel segs.
- any nontrivial rational algebraic equation
 ↳ something not universally true - depends on input
 → lower-dimensional subset of \mathbb{R}^{dn}
⇒ almost every realization is generic
i.e. random realization is generic with probability 1
 or ϵ -perturbation

another definition in L10

Rigidity is generic: for any graph, either

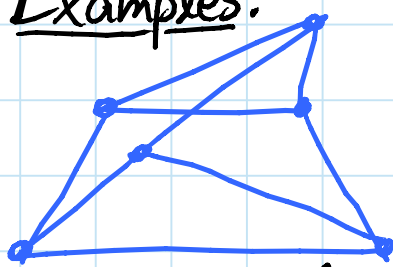
- all generic realizations are rigid

⇒ "generically rigid"

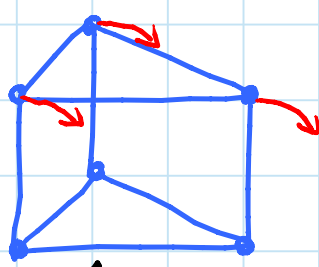
or - all generic realizations are flexible

⇒ "generically flexible"

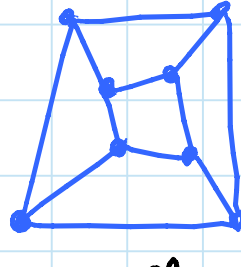
Examples:



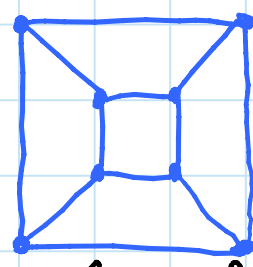
gen. rigid



rarely flex.



gen. flex.



rarely rigid

Minimally generically rigid graph = gen. rigid
+ removing any edge \Rightarrow gen. flexible

- characterize these first
- then gen. rigid graphs = supergraphs of these

Degree-of-freedom analysis: min. gen. rigid graph
in 2D has exactly $2n-3$ edges (n vertices)
& in d D has exactly $dn - \frac{d(d+1)}{2}$ edges

Proof: Empty graph has dn degrees of freedom.
Rigid graph has $\frac{d(d+1)}{2}$ degrees of freedom
for rigid motion (2 transl. + 1 rotation in 2D)

Each edge imposes one algebraic constraint,
either: = $(dn-1)$ -dim. algebraic surface

- algebraically dependent: (redundant edge)
always holds given previous constraints
 \Rightarrow configuration space unchanged

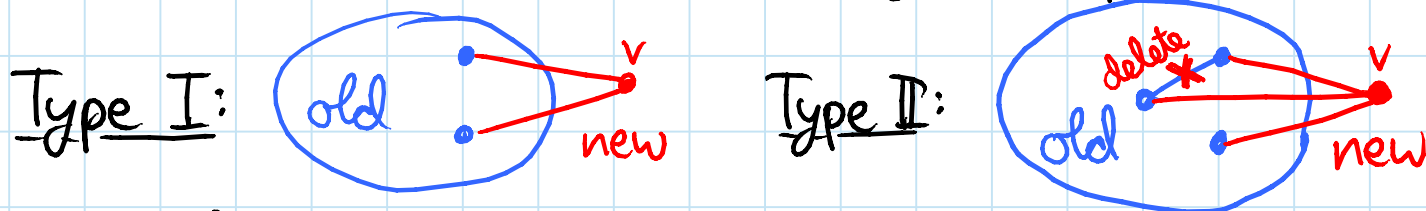
or - algebraically independent:
almost never holds (only if degenerate)
 \Rightarrow decreases dimension by 1

(in degenerate cases, can reduce
dimension by 0 or more than 1:
surfaces meeting degenerately in
fewer dimensions) \square



Henneberg characterization: [Henneberg 1911]

a graph is minimally generically rigid in 2D \Leftrightarrow it can be built from $\bullet \rightarrow \bullet$ using two operations:



Proof: (\Leftarrow) holds in any dimension \sim degrees d & $d+1$

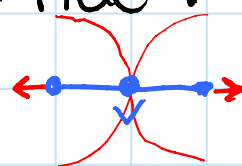
① G gen. rigid $\Leftrightarrow G'$ resulting from Type I is

(\Leftarrow) if G is gen. flexible

then v can "come for the ride":

(intersection of 2 circles)

generically v won't be taut:



(\Rightarrow) if G is gen. rigid

then so is G' : can't just move v

② G min. gen. rigid $\Leftrightarrow G'$ from Type I is

— similar: prove $G-e$ flexible for all e

$\Leftrightarrow G'-e$ flexible for all e

③ G min. gen. rigid $\Rightarrow G'$ from Type II is

— harder: need infinitesimal rigidity [L4]

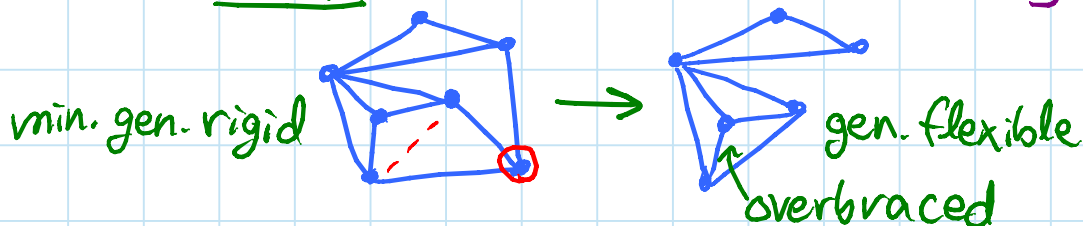
enough for (\Leftarrow) proof

"converse" of ③ \rightarrow

④ G' min. gen. rigid with degree-3 vertex v
 $\Rightarrow G'$ is result of Type II of some min. gen. rigid G

— ditto

Careful: not all work! [Abbott]



Proof of Henneberg characterization: (cont'd)

(\Rightarrow) holds only in 2D

- DOF analysis $\Rightarrow 2n-3$ edges

- Handshaking Lemma: in any graph,
 $\sum \text{vertex degrees} = 2(\# \text{edges})$

\Rightarrow average degree $= 4 - \frac{6}{n} < 4$

\Rightarrow some vertex v of degree < 4

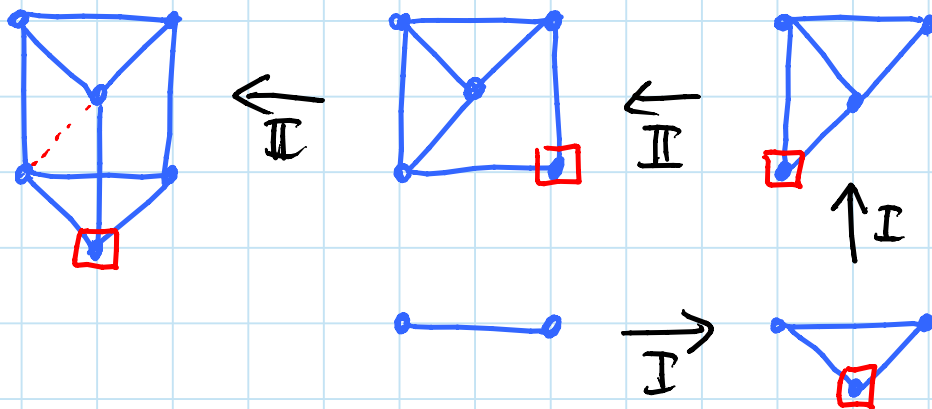
- can't be degree 1 or else flexible

- if degree 2: delete v (anti-Type I op.)

② \Rightarrow still min. gen. rigid
induct

- if degree 3: ④ \Rightarrow anti-type II op.
induct

Example:



Laman's characterization: [Laman 1970]

a graph is minimally generically rigid in 2D \Leftrightarrow
it has $2n-3$ edges ($n > 1$ vertices)
& every subset of k vertices induces $\leq 2k-3$ edges

Proof: [based on Jack Graver's *Counting on Frameworks*]

(\Rightarrow) Consider Henneberg construction.

Each operation adds one vertex ($n \nearrow 1$),

increases # edges by 2, and

any subset of vertices either

- excludes $v \Rightarrow$ # edges $\downarrow 0$ or 1
- or - includes $v \Rightarrow k \nearrow 1$, # edges $\nearrow \leq 2$

Base case: 

(\Leftarrow) $2n-3$ edges, so again

there is a vertex v of degree < 4

- can't be degree 1: then remaining $n-1$ vertices induce $2n-4$ edges

- if degree 2:

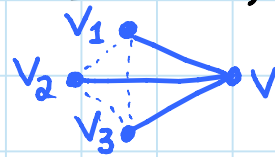
- delete v (anti-Type I op.)

- every subset of remaining vertices induces same # edges as before

\Rightarrow still Laman

- induct & use ②

Proof of Laman characterization: (cont'd)



(\Leftarrow) - if degree 3:

- some edge is missing among v 's neighbors:
4 vertices induce $\leq 2 \cdot 4 - 3 = 5$ out of 6 edges
- need to find anti-Type II op. preserving Laman
- any k vertices including v_1, v_2, v_3 & not v must have $< 2k - 3$ edges (\Rightarrow room for 1 more):
adding v increases $k \uparrow 1$, #edges $\uparrow 3$
- suppose there are k_i vertices S_i ($i=1,2,3$) containing v_{i+1}, v_{i+2} but not v & inducing $2k_i - 3$ edges
- Inclusion-Exclusion principle:

$$E(S_1 \cup S_2) \geq E(S_1) + E(S_2) - E(S_1 \cap S_2)$$

$$= 2(k_1 + k_2) - 6 - E(S_1 \cap S_2)$$



suppose > 1 vertex

edges \checkmark
in induced
subgraph

$$\geq 2(k_1 + k_2 - |S_1 \cap S_2|) - 3$$

$$= 2|S_1 \cup S_2| - 3$$

but $S_1 \cup S_2$ contains v_1, v_2, v_3 & not $v \Rightarrow$ impossible

$$\Rightarrow S_1 \cap S_2 = \{v_3\}, S_2 \cap S_3 = \{v_1\}, S_1 \cap S_3 = \{v_2\}$$

$$\Rightarrow |S_1 \cup S_2 \cup S_3| = k_1 + k_2 + k_3 - 3$$

$$\& E(S_1 \cup S_2 \cup S_3) \geq E(S_1) + E(S_2) + E(S_3)$$

$$= 2(k_1 + k_2 + k_3) - 9$$

$$= 2|S_1 \cup S_2 \cup S_3| - 3$$

but $S_1 \cup S_2 \cup S_3$ contains v_1, v_2, v_3 & not v

\Rightarrow can add some edge among v_1, v_2, v_3
& delete v & preserve Laman; induct. \square

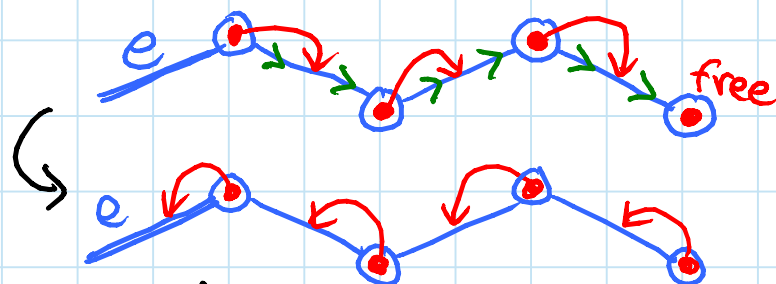
$O(nm)$ algorithm: (n vertices, m edges)
[Jacobs & Hendrickson 1997]

Edge set is independent if,
for each edge e & for each subset of k vertices,
induced edges + 3 copies of $e \leq 2k$.

Equivalent formulation of Laman: a graph is
generically rigid \Leftrightarrow has $2n-3$ independent edges
(forming minimally generically rigid subgraph)

Pebble algorithm:

- repeatedly try to add edge to independent set
(outer loop of m iterations)
- give each vertex 2 pebbles,
assignable to any 2 incident edges
- to see whether edge e can be added to indep. set:
 - add 4 copies of $e \sim$ unpebbled
 - try to get a pebble on every edge by
searching for directed path from e to free pebble:



directed graph
connectivity -
 $O(n)$ time

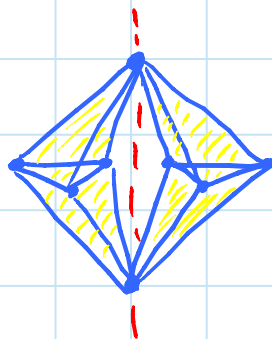
- succeed precisely if e is independent
- otherwise e is redundant: discard

OPEN: fastest 2D generic rigidity test?

- best so far: $O(n^2)$ [Hendrickson 1992]
- similar to above, using bipartite matching
- $O(n^{1.5} \sqrt{\log n})$ via decomposition into two disjoint spanning trees [Daescu & Kurdia 2009]
- unlikely to do better than matching... ($O(n^{1.5})$)
- can we use dynamic connectivity DSs?

OPEN: combinatorial test for 3D generic rigidity

- Laman condition ($3n-6$) still necessary but insufficient:



double
bananas

PROJECT: implement Laman and/or Henneberg

- color code over/underbracing, min. rigid subgraph
- show Henneberg construction

PROJECT: Henneberg computer game/puzzle

- player starts from $\bullet \rightarrow \bullet$ & tries to build target min. gen. rigid graph by Henneberg ops.
- fairly easy backwards; hard forwards!

Rigidity of polyhedra: (e.g. geodesic domes)
(see Connelly's 1993 survey)

Cauchy's Theorem: [Cauchy 1813; Steinitz 1934]

a polyhedron with rigid faces
has only one convex configuration
(modulo rigid motion)

- true even if faces are pieces of paper
(can add/remove creases) [Alexandrov 1958]

Dehn's Theorem: [Dehn 1916]

the edge skeleton of a (strictly) convex
polyhedron with triangular faces
is (infinitesimally) rigid

Alexandrov's Theorem: [Alexandrov 1958]

the edge skeleton of a triangulated
convex polyhedron is (infinitesimally) rigid

Bellows Theorem: [Connelly, Sabitov, Walz 1997]

when (nonconvex) polyhedron flexes,
its volume is preserved

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6.849 Geometric Folding Algorithms: Linkages, Origami, Polyhedra
Fall 2012

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