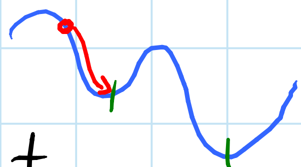
6.849	Class 13	Oct. 23, 2012

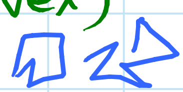
- Why expansiveness?
  - reduces "unfolding without crossing" to "infinitesimal flexibility of tensegrity"
  - also stronger mathematically ~ e.g. next lecture uses (strict) expans. to "thicken" bars into polygons

- Local minima in energy algorithm? NO
  - gradient flow $-\nabla E$ finds local minimum: stops when no downhill motion



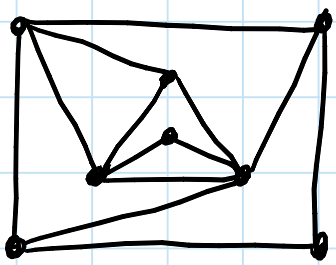
  - but Carpenter's Rule Theorem
    $\Rightarrow$ every configuration that's not already straight/convex has an expansive motion $d(u, vw) \nearrow$
    $\Rightarrow$ has energy-decreasing motion

$$E = \sum_{\text{edge } vw} \sum_{\text{vertex } u} 1/d(u, vw) \searrow$$

    $\Rightarrow$ continue until straight/convex
    (actually until ε from straight/convex)
  - if linkage has multiple components: 
    - after components fly far away, ($\approx n/\varepsilon$) contribute little to energy decrease
    $\Rightarrow$ gradient motion unfolds components (within ε of outer-convex)

- **Pointed pseudotriangulations:**
  - original use: ray shooting in polygons via "balanced" ($\Theta(\lg n)$-diameter) pseudotriang. [Chazelle, Edelsbrunner, Grigni, Guibas, Hershberger, Sharir, Snoeyink – Algorithmica 1994]
  - $2n-3$ edges & $n-2$ pseudotriangles vs. $2n-3+i$ edges & $n-2+i$ triangles in triangulation
    ↳ # interior vertices (not on convex hull) = nonpointed vertices in pseudotriangulation
  - minimally generically rigid (Laman) [Streinu 2000]
    - any (induced) subgraph is pointed
      ⇒ can be completed into pointed pseudotriang. by adding edges until maximal
    - afterword, # edges $= 2n-3$
  - every *planar* Laman graph has a pointed pseudotriangulation realization [Haas, Orden, Rote, Santos, Servatius, Servatius, Souvaine, Streinu, Whiteley – CGTA 2005]
    - via Henneberg construction

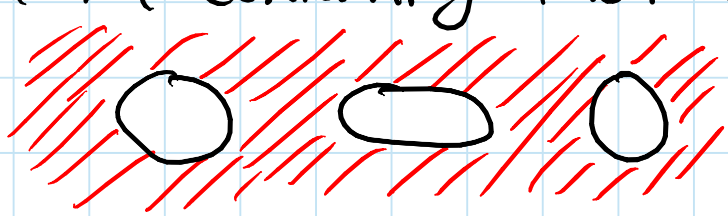
Example:



- removing convex-hull edge e ⇒ expansive
  - was Laman ⇒ now (generically) flexible
  - add all pairwise struts, including e
  - from duality, suffices to prove that all equilibrium stresses are zero on e ⇒ e (at least) can expand
  - from CDR, nonzero stresses must be on or interior to convex bar polygons (⇒ not e)
    - let M = region of xy plane lifting to maximum z coordinate
    ⇒ ∂M consists of mountains ⇒ bars
    - at vertex v of ∂M:

reflex angle between bars must be locally in M (only valleys there)

⇒ ∂M consists of convex bar polygons with M containing their exterior

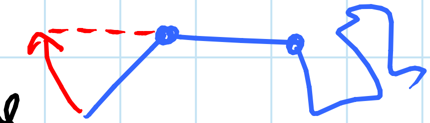


M

- pointed pseudotriangulations minus hull edge form all "extreme rays" (edges) of cone of expansive motions [Rote, Santos, Streinu 2002]

- Linear equilateral trees: [Abel, Demaine, Demaine, Eisenstat, Lynch, Schardl, Shapiro-Ellowitz — ISAAC 2011]
  - recall locked linear OR equilateral trees
  - but no locked linear equilateral tree
    - initially view as path 
    - repeatedly split at "break" & canonicalize by one move
    - ⇒ canonical with one more vertex
    - induct

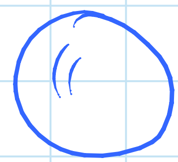
- Linkages in 4D: [Cocan & O'Rourke 2001]
  - every open chain can be straightened in 4D⁺:
    - idea: move first bar to "extend" second bar
    - then "fuse" that joint, treating first two bars as one

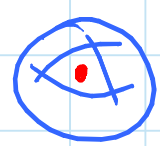


    ⇒ effectively $n-1$ bars left: induct
    - problem: goal state for first bar might intersect rest of chain

works in 3D too!

      - if so, just perturb the linkage (actually can just move the vertex to be straightened)
    - key: first bar can reach any nonobstructed position
      - configurations around joint = points on 3D sphere in 4D (centered at joint)



        (analogy: 3D chain, points on usual 2D sphere)
      - obstacle = projection of 1D bar onto sphere = 1D arc
      - deleting 1D arcs keeps 3D sphere connected
        (analogy: deleting 0D points from usual 2D sphere)
        [in 3D, could build a "cage": ]
  - every tree can be flattened (similar technique)
  - every cycle can be convexified (diff. approach)

6.849 Geometric Folding Algorithms: Linkages, Origami, Polyhedra
Fall 2012