6.231 DYNAMIC PROGRAMMING

LECTURE 22

LECTURE OUTLINE

- Aggregation as an approximation methodology
- Aggregate problem
- Examples of aggregation
- Simulation-based aggregation
- Q-Learning

PROBLEM APPROXIMATION - AGGREGATION

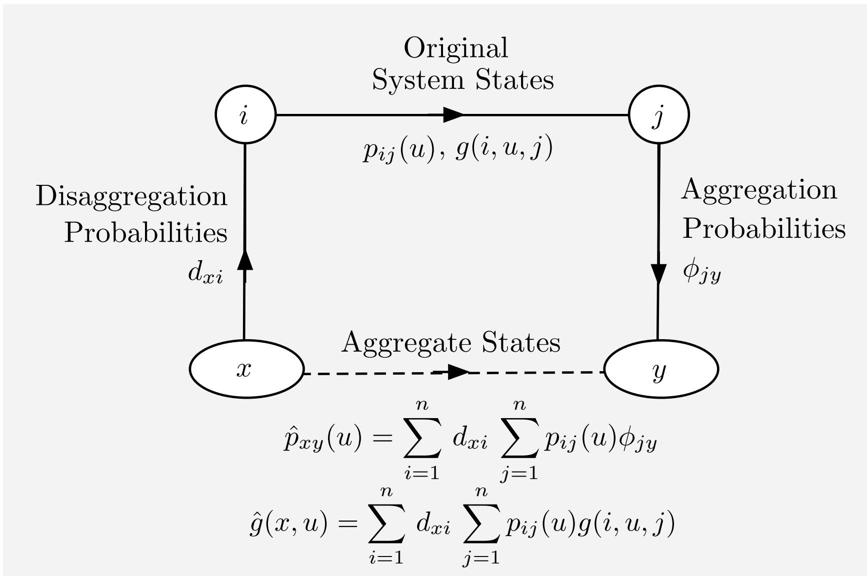
- Another major idea in ADP is to approximate the cost-to-go function of the problem with the cost-to-go function of a simpler problem. The simplification is often ad-hoc/problem dependent.

- Aggregation is a systematic approach for problem approximation. Main elements:
  - Introduce a few "aggregate" states, viewed as the states of an "aggregate" system
  - Define transition probabilities and costs of the aggregate system, by relating original system states with aggregate states
  - Solve (exactly or approximately) the "aggregate" problem by any kind of value or policy iteration method (including simulation-based methods)
  - Use the optimal cost of the aggregate problem to approximate the optimal cost of the original problem

- Hard aggregation example: Aggregate states are subsets of original system states, treated as if they all have the same cost.

AGGREGATION/DISAGGREGATION PROBS

- The aggregate system transition probabilities are defined via two (somewhat arbitrary) choices



- For each original system state j and aggregate state y, the aggregation probability ϕ_{jy}
  - The "degree of membership of j in the aggregate state y."
  - In hard aggregation, $\phi_{jy} = 1$ if state j belongs to aggregate state/subset y.
- For each aggregate state x and original system state i, the disaggregation probability d_{xi}
  - The "degree of i being representative of x."
  - In hard aggregation, one possibility is all states i that belongs to aggregate state/subset x have equal d_{xi}.

AGGREGATE PROBLEM

- The transition probability from aggregate state x to aggregate state y under control u

$$\hat{p}_{xy}(u) = \sum_{i=1}^{n} d_{xi} \sum_{j=1}^{n} p_{ij}(u)\phi_{jy}, \quad \text{or } \hat{P}(u) = DP(u)\Phi$$

where the rows of D and Φ are the disaggr. and aggr. probs.

- The aggregate expected transition cost is

$$\hat{g}(x,u) = \sum_{i=1}^{n} d_{xi} \sum_{j=1}^{n} p_{ij}(u) g(i,u,j), \quad \text{or } \hat{g} = DPg$$

- The optimal cost function of the aggregate problem, denoted $\hat{R}$, is

$$\hat{R}(x) = \min_{u \in U} \left[\hat{g}(x,u) + \alpha \sum_{y} \hat{p}_{xy}(u) \hat{R}(y) \right], \qquad \forall\, x$$

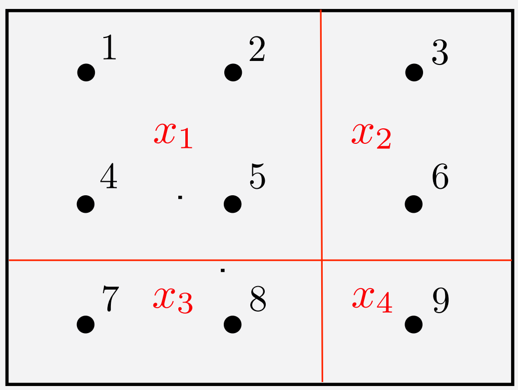
or $\hat{R} = \min_u [\hat{g} + \alpha \hat{P} \hat{R}]$ - Bellman's equation for the aggregate problem.

- The optimal cost J^* of the original problem is approximated using interpolation, $J^* \approx \tilde{J} = \Phi \hat{R}$:

$$\tilde{J}(j) = \sum_{y} \phi_{jy} \hat{R}(y), \qquad \forall\, j$$

EXAMPLE I: HARD AGGREGATION

- Group the original system states into subsets, and view each subset as an aggregate state
- Aggregation probs: $\phi_{jy} = 1$ if j belongs to aggregate state y.

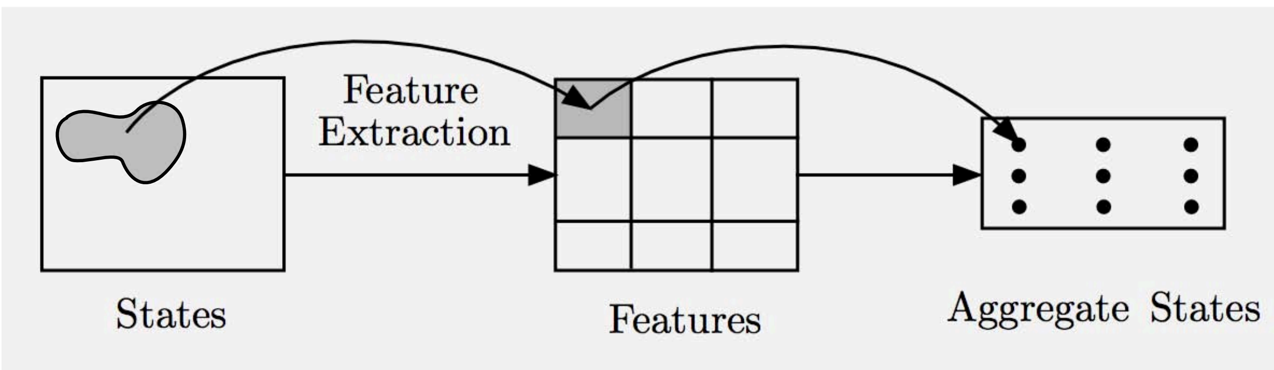


$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Disaggregation probs: There are many possibilities, e.g., all states i within aggregate state x have equal prob. d_{xi}.
- If optimal cost vector J^* is piecewise constant over the aggregate states/subsets, hard aggregation is exact. Suggests grouping states with “roughly equal” cost into aggregates.
- Soft aggregation (provides “soft boundaries” between aggregate states).

EXAMPLE II: FEATURE-BASED AGGREGATION

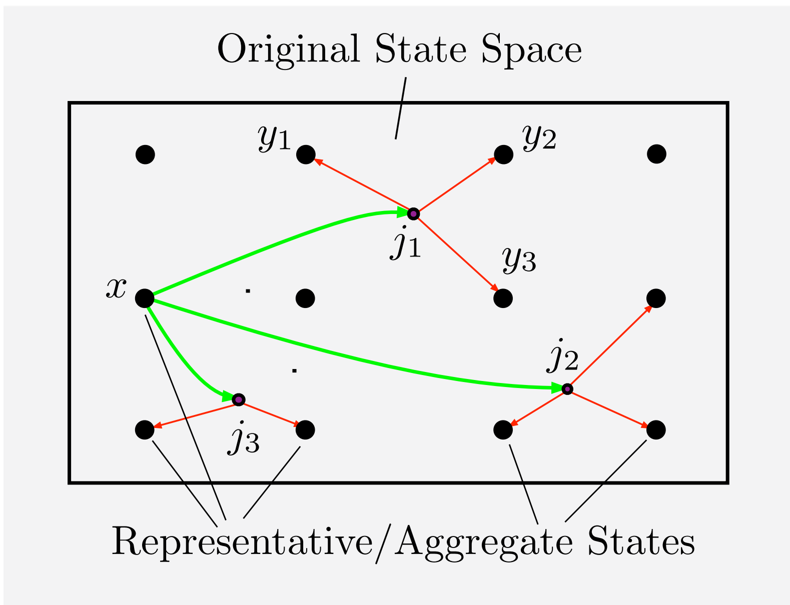
- If we know good features, it makes sense to group together states that have "similar features"
- Essentially discretize the features and assign a weight to each discretization point



- A general approach for passing from a feature-based state representation to an aggregation-based architecture
- Hard aggregation architecture based on features is more powerful (nonlinear/piecewise constant in the features, rather than linear)
- ... but may require many more aggregate states to reach the same level of performance as the corresponding linear feature-based architecture

EXAMPLE III: REP. STATES/COARSE GRID

- Choose a collection of “representative” original system states, and associate each one of them with an aggregate state. Then “interpolate”



- Disaggregation probs. are $d_{xi} = 1$ if i is equal to representative state x.

- Aggregation probs. associate original system states with convex combinations of rep. states

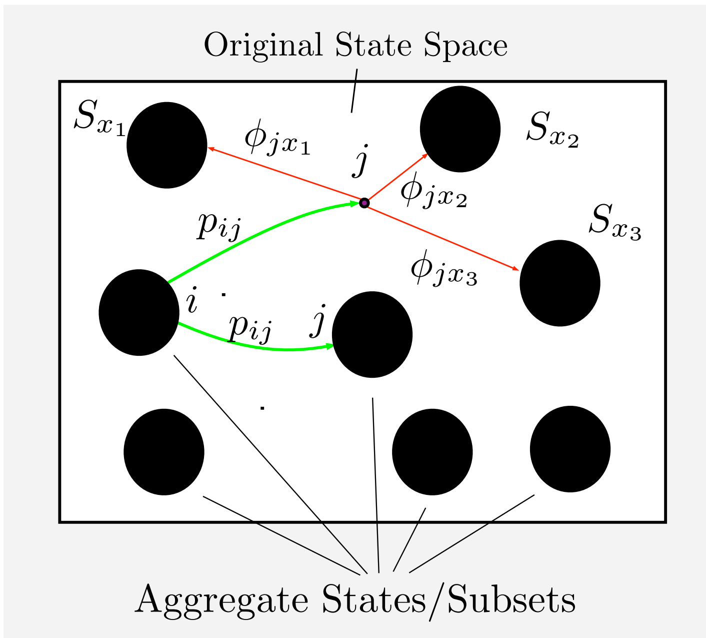
$$j \sim \sum_{y \in \mathcal{A}} \phi_{jy} y$$

- Well-suited for Euclidean space discretization

- Extends nicely to continuous state space, including belief space of POMDP

EXAMPLE IV: REPRESENTATIVE FEATURES

- Choose a collection of “representative” subsets of original system states, and associate each one of them with an aggregate state



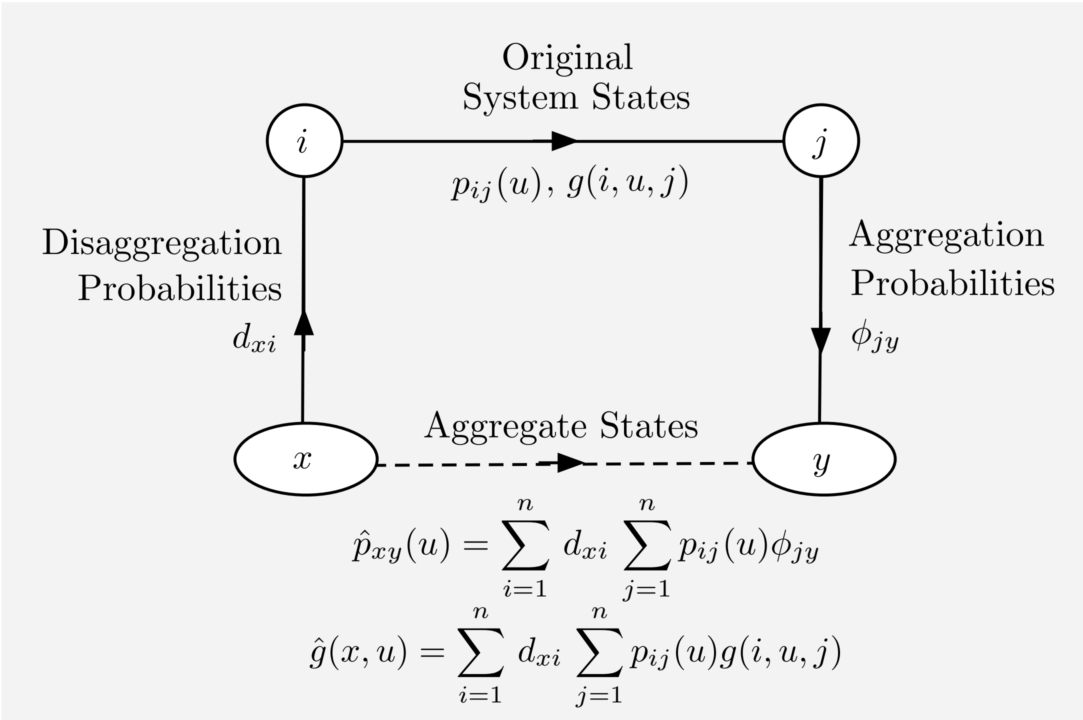
- Common case: S_x is a group of states with “similar features”

- Hard aggregation is special case: $\cup_x S_x = \{1, \ldots, n\}$

- Aggregation with representative states is special case: S_x consists of just one state

- With rep. features, aggregation approach is a special case of projected equation approach with “seminorm” projection. So the TD methods and multistage Bellman Eq. methodology apply

APPROXIMATE PI BY AGGREGATION



- Consider approximate PI for the original problem, with evaluation done using the aggregate problem (other possibilities exist - see the text)

- Evaluation of policy μ: $\tilde{J} = \Phi R$, where $R = DT_\mu(\Phi R)$ (R is the vector of costs of aggregate states corresponding to μ). May use simulation.

- Similar form to the projected equation $\Phi R = \Pi T_\mu(\Phi R)$ (ΦD in place of Π).

- Advantages: It has no problem with exploration or with oscillations.

- Disadvantage: The rows of D and Φ must be probability distributions.

Q-LEARNING I

- Q-learning has two motivations:
  - Dealing with multiple policies simultaneously
  - Using a model-free approach [no need to know $p_{ij}(u)$, only be able to simulate them]
- The Q-factors are defined by

$$Q^*(i,u) = \sum_{j=1}^{n} p_{ij}(u)\big(g(i,u,j) + \alpha J^*(j)\big), \quad \forall\ (i,u)$$

- Since $J^* = TJ^*$, we have $J^*(i) = \min_{u\in U(i)} Q^*(i,u)$ so the Q factors solve the equation

$$Q^*(i,u) = \sum_{j=1}^{n} p_{ij}(u)\left(g(i,u,j) + \alpha \min_{u'\in U(j)} Q^*(j,u')\right)$$

- $Q^*(i,u)$ can be shown to be the unique solution of this equation. Reason: This is Bellman's equation for a system whose states are the original states $1,\ldots,n$, together with all the pairs (i,u).

- Value iteration: For all (i,u)

$$Q(i,u) := \sum_{j=1}^{n} p_{ij}(u)\left(g(i,u,j) + \alpha \min_{u'\in U(j)} Q(j,u')\right)$$

Q-LEARNING II

- Use some randomization to generate sequence of pairs (i_k, u_k) [all pairs (i, u) are chosen infinitely often]. For each k, select j_k according to $p_{i_k j}(u_k)$.

- Q-learning algorithm: updates $Q(i_k, u_k)$ by

$$Q(i_k, u_k) := \big(1 - \gamma_k(i_k, u_k)\big) Q(i_k, u_k) + \gamma_k(i_k, u_k) \left(g(i_k, u_k, j_k) + \alpha \min_{u' \in U(j_k)} Q(j_k, u') \right)$$

- Stepsize $\gamma_k(i_k, u_k)$ must converge to 0 at proper rate (e.g., like $1/k$).

- Important mathematical point: In the Q-factor version of Bellman's equation the order of expectation and minimization is reversed relative to the ordinary cost version of Bellman's equation:

$$J^*(i) = \min_{u \in U(i)} \sum_{j=1}^{n} p_{ij}(u) \big(g(i, u, j) + \alpha J^*(j) \big)$$

- Q-learning can be shown to converge to true/exact Q-factors (sophisticated stoch. approximation proof).

- Major drawback: Large number of pairs (i, u) - no function approximation is used.

Q-FACTOR APPROXIMATIONS

- Basis function approximation for Q-factors:

$$\tilde{Q}(i,u,r) = \phi(i,u)'r$$

- We can use approximate policy iteration and LSPE/LSTD/TD for policy evaluation (exploration issue is acute).

- Optimistic policy iteration methods are frequently used on a heuristic basis.

- Example (very optimistic). At iteration k, given r_k and state/control (i_k, u_k):

  (1) Simulate next transition (i_k, i_{k+1}) using the transition probabilities $p_{i_k j}(u_k)$.

  (2) Generate control u_{k+1} from

$$u_{k+1} = \arg \min_{u \in U(i_{k+1})} \tilde{Q}(i_{k+1}, u, r_k)$$

  (3) Update the parameter vector via

$$r_{k+1} = r_k - (\text{LSPE or TD-like correction})$$

- Unclear validity. Solid basis for aggregation case, and for case of optimal stopping (see text).