CHAPTER 3

FLOW PAST A SPHERE II: STOKES' LAW, THE BERNOULLI EQUATION, TURBULENCE, BOUNDARY LAYERS, FLOW SEPARATION

INTRODUCTION

1 So far we have been able to cover a lot of ground with a minimum of material on fluid flow. At this point I need to present to you some more topics in fluid dynamics—inviscid fluid flow, the Bernoulli equation, turbulence, boundary layers, and flow separation—before returning to flow past spheres. This material also provides much of the necessary background for discussion of many of the topics on sediment movement to be covered in Part II. But first we will make a start on the nature of flow of a viscous fluid past a sphere.

THE NAVIER-STOKES EQUATION

2 The idea of an equation of motion for a viscous fluid was introduced in the Chapter 2. It is worthwhile to pursue the nature of this equation a little further at this point. Such an equation, when the forces acting in or on the fluid are those of viscosity, gravity, and pressure, is called the ***Navier–Stokes equation***, after two of the great applied mathematicians of the nineteenth century who independently derived it.

3 It does not serve our purposes to write out the Navier–Stokes equation in full detail. Suffice it to say that it is a *vector partial differential equation*. (By that I mean that the force and acceleration terms are vectors, not scalars, and the various terms involve partial derivatives, which are easy to understand if you already know about differentiation.) The single vector equation can just as well be written as three scalar equations, one for each of the three coordinate directions; this just corresponds to the fact that a force, like any vector, can be described by its scalar components in the three coordinate directions.

4 The Navier–Stokes equation is notoriously difficult to solve in a given flow problem to obtain spatial distributions of velocities and pressures and shear stresses. Basically the reasons are that the acceleration term is nonlinear, meaning that it involves products of partial derivatives, and the viscous-force term contains second derivatives, that is, derivatives of derivatives. Only in certain special situations, in which one or both of these terms can be simplified or neglected, can the Navier–Stokes equation be solved analytically. But numerical solutions of the full Navier–Stokes equation are feasible for a much wider range of flow problems, now that computers are so powerful.

FLOW PAST A SPHERE AT LOW REYNOLDS NUMBERS

5 We will make a start on the flow patterns and fluid forces associated with flow of a viscous fluid past a sphere by restricting consideration to low Reynolds numbers $\rho UD/\mu$ (where, as before, U is the uniform approach velocity and D is the diameter of the sphere).

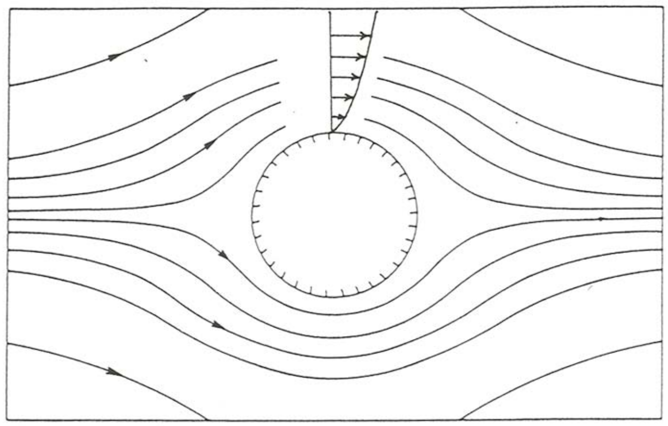

Figure 3-1. Steady flow of a viscous fluid at very low Reynolds numbers ("creeping flow") past a sphere. The flow lines are shown in a planar section parallel to the flow direction and passing through the center of the sphere.

6 At very low Reynolds numbers, Re << 1, the flow lines relative to the sphere are about as shown in Figure 3-1. The first thing to note is that for these very small Reynolds numbers the flow pattern is symmetrical front to back. The flow lines are straight and uniform in the free stream far in front of the sphere, but they are deflected as they pass around the sphere. For a large distance away from the sphere the flow lines become somewhat more widely spaced, indicating that the fluid velocity is less than the free-stream velocity. Does that do damage to your intuition? One might have guessed that the flow lines would be more crowded together around the midsection of the sphere, reflecting a greater velocity instead—and as will be shown later in this chapter, that is indeed the case at much higher Reynolds numbers. (See a later section for more on what I have casually called flow lines here.) For very low Reynolds numbers, however, the effect of "crowding", which acts to increase the velocity, is more than offset by the effect of viscous retardation, which acts to decrease the velocity.

7 The velocity of the fluid is everywhere zero at the sphere surface (remember the no-slip condition) and increases only slowly away from the sphere, even in the vicinity of the midsection: at low Reynolds numbers, the retarding effect of the sphere is felt for great distances out into the fluid. You will see later in this chapter that the zone of retardation shrinks greatly as the Reynolds number increases, and the "crowding" effect causes the velocity around the midsection of the sphere to be greater than the free-stream velocity except very near the surface of the sphere; more on that later.

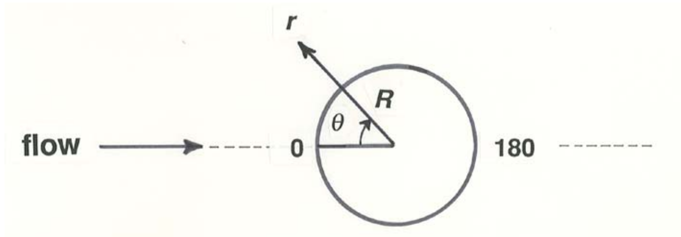

Figure 3-2. Coordinates for description of the theoretical distribution of velocity in flow past a sphere at very low Reynolds numbers (creeping flow).

8 If you would like to see for yourself how the velocity varies in the vicinity of the sphere, Equations 3.1 give the theoretical distribution of velocity v, as a function of distance r from the center of the sphere and the angle θ measured around the sphere from 0° at the front point to 180° at the rear point (Figure 3-2):

$$u_r = -U\cos\theta\left(1 - \frac{3R}{2r} + \frac{R^3}{2r^3}\right)$$

$$u\theta = U\sin\theta\left(1 - \frac{3R}{4r} - \frac{R^3}{4r^3}\right) \qquad (3.1)$$

This result was obtained by Stokes (1851) by specializing the Navier–Stokes equations for an approaching flow that is so slow that accelerations of the fluid as it passes around the sphere can be ignored, resulting in an equation that can be solved analytically. I said in Chapter 2 that fluid density ρ is needed as a variable to describe the drag force on a sphere because accelerations are produced in the fluid as the sphere moves through it. If these accelerations are small enough, however, it is reasonable to expect that their effect on the flow and forces can be neglected. Flows of this kind are picturesquely called ***creeping flows***. The reason, to which I alluded in the previous section, is that in the Navier–Stokes equations the term for rate of change of momentum becomes small faster than the two remaining terms, for viscous forces and pressure forces, as the Reynolds number decreases.

9 You can see from Equations 3.1 that as $r \to \infty$ the velocity approaches its free-stream magnitude and direction. The $1/r$ dependence in the second terms in the parentheses on the right-hand sides of Equations 3.1 reflects the appreciable distance away from the sphere the effects of viscous retardation are felt. A simple computation using Equations 3.1 shows that, at a distance equal to the sphere diameter from the surface of the sphere at the midsection in the direction normal to the free-stream flow, the velocity is still only 50% of the free-stream value.

10 At every point on the surface of the sphere there is a definite value of fluid pressure (normal force per unit area) and of viscous shear stress (tangential force per unit area). These values also come from Stokes' solution for creeping flow around a sphere. For the shear stress, you could use Equations 3.1 to find the velocity gradient at the sphere surface and then use Equation 1.9 to find the shear stress. For the pressure, Stokes found a separate equation,

$$p - p_0 = \frac{3}{2}\frac{\mu UR}{r^2}\cos\theta \tag{3.2}$$

where p_o is the free-stream pressure. Figures 3-3 and 3-4 give an idea of the distribution of these forces. It is easy to understand why the viscous shear stress should be greatest around the midsection and least on the front and back surface of the sphere, because that is where the velocity near the surface of the sphere is greatest. The distribution of pressure, high in the front and low in the back, also makes intuitive sense. It is interesting, though, that there is a large front-to-back difference in pressure despite the nearly perfect front-to-back symmetry of the flow.

11 You can imagine adding up both pressures and viscous shear stresses over the entire surface, remembering that both magnitude and direction must be taken into account, to obtain a resultant pressure force and a resultant viscous force on the sphere. Because of the symmetry of the flow, both of these resultant forces are directed straight downstream. You can then add them together to obtain a grand resultant, the total drag force F_D. Using the solutions for velocity and pressure given above (Equations 3.1 and 3.2), Stokes obtained the result

$$F_D = 6\pi\mu UR \tag{3.3}$$

for the total drag force on the sphere. Density does not appear in Stokes' law because it enters the equation of motion only the mass-time-acceleration term, which was neglected. For Reynolds numbers less than about one, the result expressed by Equation 3.3, called ***Stokes' law***, is in nearly perfect agreement with experiment. It turns out that in the Stokes range, for Re << 1, exactly one-third of F_D is due to the pressure force and two-thirds is due to the viscous force.

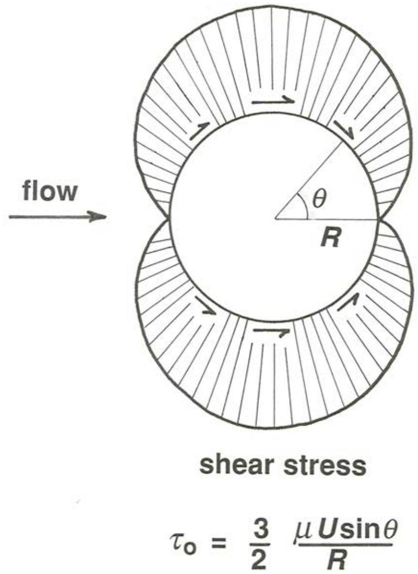

Figure 3-3. Distribution of shear stress on the surface of a sphere in a flow of viscous fluid at very low Reynolds numbers (creeping flow). The distribution is shown in a planar section parallel to the flow direction and passing through the center of the sphere.

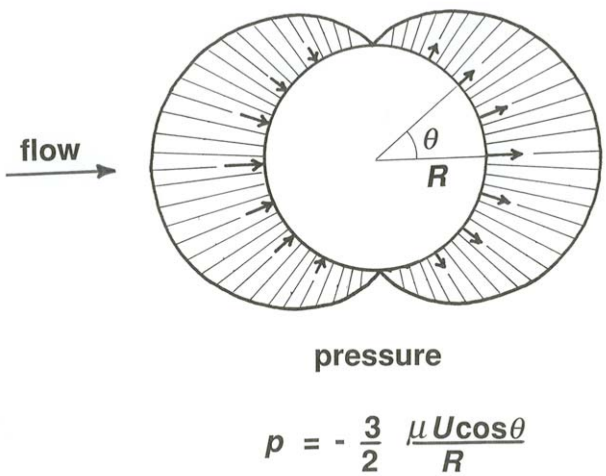

Figure 3-4. Distribution of pressure on the surface of a sphere in a flow of viscous fluid at very low Reynolds number (creeping flow). The distribution is shown in a planar section parallel to the flow direction and passing through the center of the sphere.

12 Now pretend that you do not know anything about Stokes' law for the drag on a sphere at very low Reynolds numbers. If you reason, as discussed above, that ρ can safely be omitted from the list of variables that influence the drag force, then you are left with four variables: F_D, U, D, and μ. The functional relationship among these four variables is necessarily

$$f(F_D, U, D, \mu) = \text{const} \tag{3.4}$$

You can form only one dimensionless variable out of the four variables F_D, U, D, and μ, namely $F_D/\mu UD$. So, in dimensionless form, the functional relationship in Equation 3.4 becomes

$$\frac{F_D}{\mu UD} = \text{const} \tag{3.5}$$

You can think of Equation 3.5 as a special case of Equation 2.2. If you massage Stokes' law (Equation 3.3) just a bit, by dividing both sides of the equation by μUR to make the equation dimensionless, and using the diameter D instead of the radius R, you obtain

$$\frac{F_D}{\mu UD} = 3\pi \tag{3.6}$$

Compare this with Equation 3.5 above. You see that dimensional analysis alone, without recourse to attempting exact solutions, provides the equation to within the proportionality constant. Stokes' theory provides the value of the constant.

13 The flow pattern around the sphere and the fluid forces that act on the sphere gradually become different as the Reynolds number is increased. The progressive changes in flow pattern with increasing Reynolds number are discussed in more detail later in this chapter, after quite a bit of necessary further background in the fundamentals of fluid dynamics.

INVISCID FLOW

14 Over the past hundred and fifty years a vast body of mathematical analysis has been devoted to a kind of fluid that exists only in the imagination: an ***inviscid fluid***, in which no viscous forces act. This fiction (in reality there is no such thing as an inviscid fluid) allows a level of mathematical progress not possible for viscous flows, because the viscous-force term in the Navier–Stokes equation disappears, and the equation becomes more tractable. The major outlines of mathematical analysis of the resulting simplified equation, which is mostly beyond the scope of these notes, were well worked out by late in the 1800s. Since then, fluid dynamicists have been extending the results and applying or specializing them to problems of interest in a great many fields.

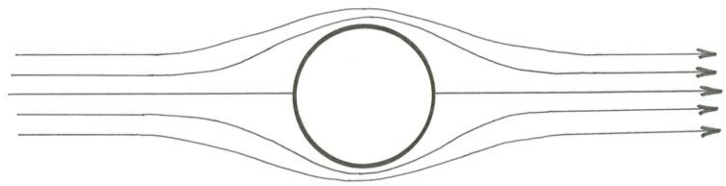

Figure 3-5. Flow of an inviscid fluid past a sphere. The flow lines are shown in a planar section parallel to the flow direction and passing through the center of the sphere.

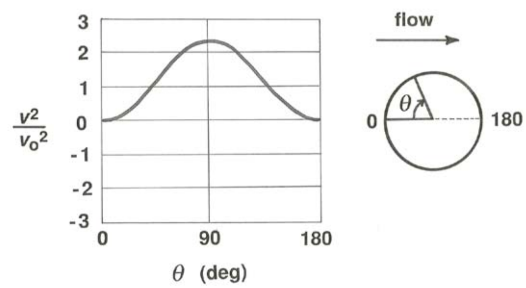

Figure 3-6. Plot of fluid velocity at the surface of a sphere that is held fixed in steady inviscid flow. The velocity, nondimensionalized by dividing by the stagnation velocity, is plotted as a function of the angle θ between the center of the sphere and points along the intersection of the sphere surface with a plane parallel to the flow direction and passing through the center of the sphere. The angle θ varies from zero at the front stagnation point of the sphere to 180° at the rear stagnation point.

15 The pattern of inviscid flow around a sphere, obtained as noted above by solving the equation of motion for inviscid flow, is shown in Figure 3-5. The arrangement of flow lines differs significantly from that in creeping viscous flow around the sphere (Figure 3-1): the symmetry is qualitatively the same, but, in contrast to creeping flow, the flow lines become more closely spaced around the midsection, reflecting acceleration and then deceleration of the flow as it passes around the sphere. Figure 3-6 is a plot of fluid velocity along the particular flow line that meets the sphere at its front point, passes back along the surface of the sphere, and leaves the sphere again at the rear point. The velocity varies symmetrically with respect to the midsection of the sphere: it falls to zero at the front point, accelerates to a maximum at the midsection, falls to zero again at the rear point, and then attains its original value again downstream. The front and rear points are called ***stagnation points***, because the fluid velocity is zero there. Note that elsewhere the velocity is *not* zero on the surface of the sphere, as it is in viscous flow. Do not let this unrealistic finite velocity on the surface of the sphere bother you; it is a consequence of the unrealistic assumption that viscous effects are absent, so that the no-slip condition is not applicable.

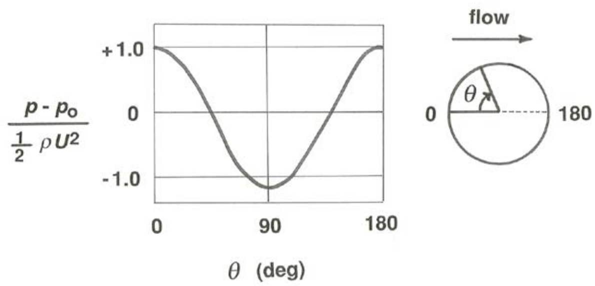

Figure 3-7. Plot of fluid pressure at the surface of a sphere that is held fixed in steady inviscid flow. The pressure relative to the stagnation pressure, nondimensionalized by dividing by $(1/2)\rho U^2$, where U is the free-stream velocity, is plotted using the same coordinates as in Figure 3-6.

16 Figure 3-7 shows the distribution of fluid pressure around the surface of a sphere moving relative to an inviscid fluid. As with velocity, pressure is distributed symmetrically with respect to the midsection, and its variation is just the inverse of that of the velocity: relative to the uniform pressure far away from the sphere, it is greatest at the stagnation points and least at the midsection. One seemingly ridiculous consequence of this symmetrical distribution is that the flow exerts no net pressure force on the sphere, and therefore, because there are no viscous forces either, it exerts no resultant force on the sphere at all! This is in striking contrast to the result noted above for creeping viscous flow past a sphere (Figure 3-3), in which the distribution of pressure on the surface of the sphere shows a strong front-to-back asymmetry; it is this uneven distribution of pressure, together with the existence of viscous shear forces on the boundary, that gives rise to the drag force on a sphere in viscous flow.

17 So the distributions of velocity and pressure in inviscid flow around a sphere, and therefore of the fluid forces on the sphere, are grossly different from the case of flow of viscous fluid around the sphere. Then what is the value of the inviscid approach? You will see in the section on flow separation later on that at higher real-fluid velocities the boundary layer in which viscous effects are concentrated next to the surface of the sphere is thin, and outside this thin layer the flow patterns and the distributions of both velocity and pressure are approximately as given by the inviscid theory. Moreover, the boundary layer is so thin for high flow velocities that the pressure on the surface of the sphere is approximately the same as that given by the inviscid solution just outside the boundary layer. And because at these high velocities the pressure forces are the main determinant of the total drag force, the inviscid approach is useful in dealing with forces on the sphere after all. Behind the sphere the flow patterns given by inviscid theory are grossly different from the real pattern at high Reynolds numbers, but you will see that one of the advantages of the inviscid assumption is that it aids in a rational explanation for the existence of this great difference.

18 In many kinds of flows around well streamlined bodies like airplane wings, agreement between the real viscous case and the ideal inviscid case is much better than for flow around blunt or bluff bodies like spheres. In flow of air around an airplane wing, viscous forces are important only in a very thin layer immediately adjacent to the wing, and outside that layer the pressure and velocity are almost exactly as given by inviscid theory (Figure 3-8). It is these inviscid solutions that allow prediction of the lift on the airplane wing: although drag on the wing is governed largely by viscous effects within the boundary layer, lift is largely dependent upon the inviscid distribution of pressure that holds just outside the boundary layer. To some extent this is true also for flow around blunt objects resting on a planar surface, like sand grains on a sand bed under moving air or water.

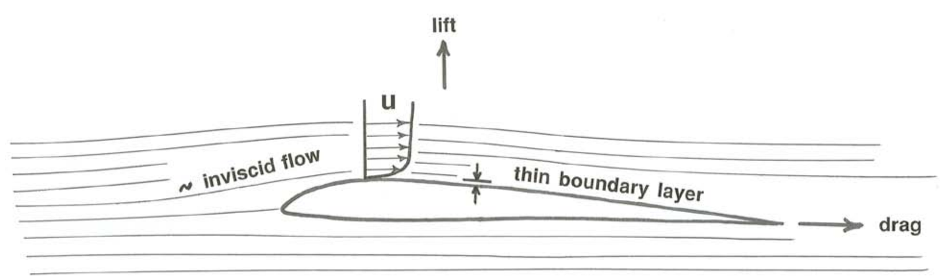

Figure 3-8. Flow of a real fluid past an airfoil, showing an overall flow pattern almost identical to that of an inviscid flow except very near the surface of the airfoil, where a thin boundary layer of retarded fluid is developed. Note that the velocity goes to zero at the surface of the airfoil.

THE BERNOULLI EQUATION

19 In the example of inviscid flow past a sphere described in the preceding section, the pressure is high at points where the velocity is low, and vice versa. It is not difficult to derive an equation, called the ***Bernoulli equation***, that accounts for this relationship. Because this will be useful later on, I will show you here how it comes about.

20 First I have to be more specific about what I have casually been calling flow lines. Fluid velocity is a vector quantity, and, because the fluid behaves as a continuum, a velocity vector can be associated with every point in the flow. (Mathematically, this is described as a *vector field*.) Continuous and smooth curves that can be drawn to be everywhere tangent to the velocity vectors throughout the vector field are called ***streamlines*** (Figure 3-9). One and only one streamline passes through each point in the flow, and at any given time there is only one such set of curves in the flow. There obviously is an infinity of streamlines passing through any region of flow, no matter how small; usually only a few representative streamlines are shown in sketches and diagrams. An important property of streamlines follows directly from their definition: the flow can never cross streamlines.

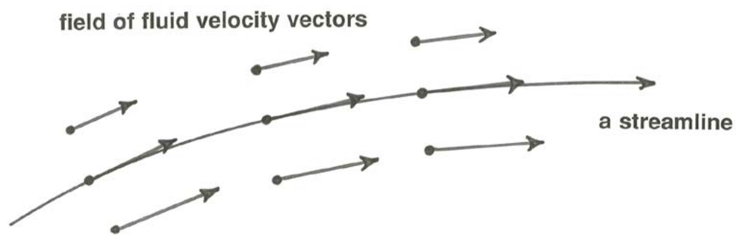

Figure 3-9. Streamlines.

21 If the flow is steady, the streamline pattern does not change with time; if the streamline pattern changes with time, the flow is unsteady. But note that the converse of each of these statements is not necessarily true, because an unsteady flow can exhibit an unchanging pattern of streamlines as velocities everywhere increase or decrease with time.

22 There are two other kinds of flow lines, with which you should not confuse streamlines (Figure 3-10): ***pathlines***, which are the trajectories traced out by individual tiny marker particles emitted from some point within the flow that is fixed relative to the stationary boundaries of the flow, and ***streaklines***, which are the streaks formed by a whole stream of tiny marker particles being emitted continuously from some point within the flow that is fixed relative to the stationary boundaries of the flow. In steady flow, streamlines and pathlines and streaklines are all the same; in unsteady flow, they are generally all different.

23 You also can imagine a tube-like surface formed by streamlines, called a ***stream tube***, passing through some region (Figure 3-11). This surface or set of streamlines can be viewed as functioning as if it were a real tube or conduit, in that there is flow through the tube but there is no flow either inward or outward across its surface.

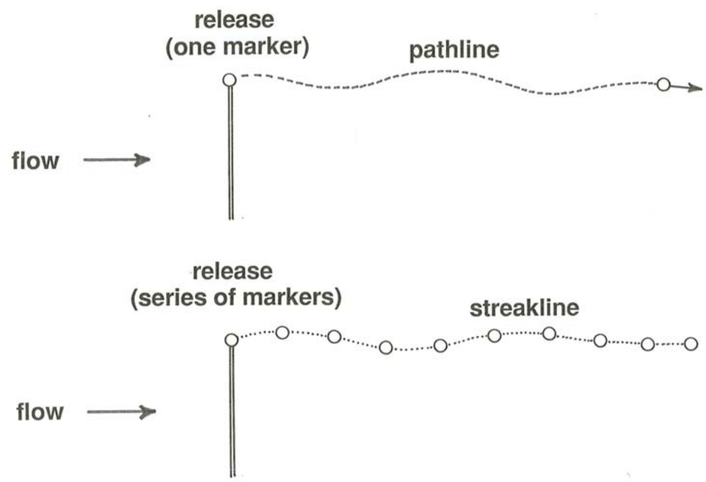

Figure 3-10. Streaklines and pathlines.

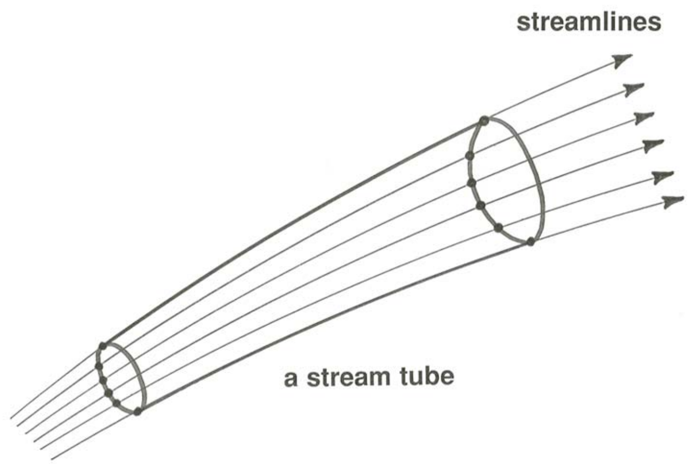

Figure 3-11. A streamtube.

24 Consider a short segment of one such tiny stream tube in a flow of incompressible fluid (Figure 3-12). Write the equation of motion (Newton's second law) for the fluid contained at some instant in this stream-tube segment. The cross-sectional area of the tube is ΔA, and the length of the segment is Δs. If the pressure at cross section 1, at the left-hand end of the segment, is p, then the force exerted on this end of the segment is $p\Delta A$. It is not important that the area of the cross section might be slightly different at the two ends (if the flow is expanding or contracting), or that p might vary slightly over the cross section, because you can make the cross-sectional area of the stream tube as small as you please. What is the force on the other end of the tube? The pressure at cross section 2 is different from that at cross section 1 by $(\partial p/\partial s)\Delta s$, the rate of change of pressure in the flow direction times the distance between the two cross sections, so the force on the right-hand end of the tube is

$$\left(p + \frac{\partial p}{\partial s}\Delta s\right) \Delta A \tag{3.7}$$

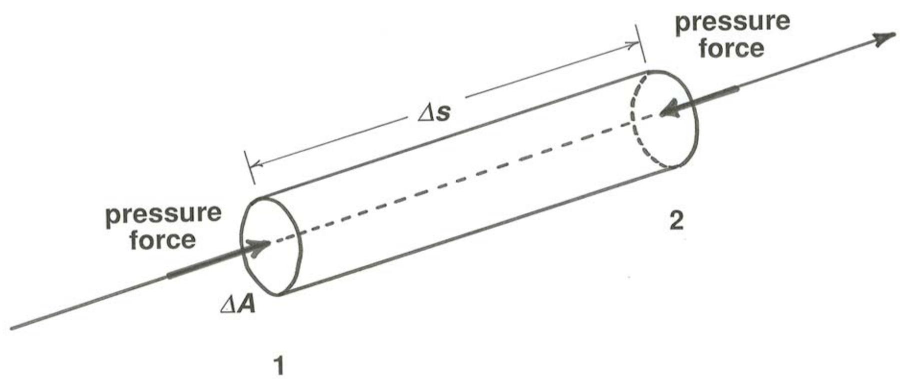

Figure 3-12. Definition sketch for derivation of the Bernoulli equation for incompressible inviscid flow.

The net force on the stream tube in the flow direction is then

$$p\Delta A - (\mathrm{p} + \frac{\partial p}{\partial s}\ \Delta s)\,\Delta A = -\frac{\partial p}{\partial s}\Delta s\Delta A \tag{3.8}$$

The pressure on the lateral surface of the tube is of no concern, because the pressure force on it acts normal to the flow direction.

25 Newton's second law, $F = d(mv)/dt$, for the fluid in the segment of the stream tube, where v is the velocity of the fluid at any point (in this section v is used not as the component of velocity in the y direction but as the component of velocity tangent to the streamline at a given point), is then

$$-\frac{\partial p}{\partial s}\ \Delta s\Delta A = \frac{d}{dt}\ [v(\rho\Delta s\Delta A)] \tag{3.9}$$

Simplifying Equation 3.9 and making use of the fact that ρ is constant and so can be moved outside the derivative,

$$-\frac{\partial p}{\partial s} = \rho \frac{dv}{dt} \tag{3.10}$$

The derivative on the right side of Equation 3.10 can be put into more convenient form by use of the chain rule and a simple "undifferentiation" of one of the resulting terms:

$$\begin{aligned} -\frac{\partial p}{\partial s} &= \rho \frac{dv}{dt} \\ &= \rho \left[\frac{\partial v}{\partial t}\frac{dt}{dt} + \frac{\partial v}{\partial s}\frac{ds}{dt}\right] \\ &= \rho \left[\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial s}\right] \\ &= \rho \left[\frac{\partial v}{\partial t} + \frac{1}{2}\frac{\partial (v^2)}{\partial s}\right] \end{aligned} \tag{3.11}$$

Equation 3.11 is strictly true only for the single streamline to which the stream tube collapses as we let ΔA go to 0, because only then need we not worry about possible variation of either p or v over the cross sections. Assuming further that the flow is steady, $\partial v/\partial t = 0$, and Equation 3.11 becomes

$$-\frac{\partial p}{\partial s} = \frac{\rho}{2}\frac{\partial (v^2)}{\partial s} \tag{3.12}$$

26 It is easy to integrate Equation (3.12) between two points 1 and 2 on the streamline (remember that this equation holds for any streamline in the flow):

$$-\int_1^2 \frac{\partial p}{\partial s}\, ds = \frac{\rho}{2}\int_1^2 \frac{\partial (v^2)}{\partial s}\, ds$$

$$p_2 - p_1 = -\frac{\rho}{2}(v_2^2 - v_1^2) \tag{3.13}$$

or, viewed in another way,

$$p + \frac{\rho v^2}{2} = \text{const} \tag{3.14}$$

You can see from Equation 3.13 that if the flow is steady and incompressible there is an inverse relationship between fluid pressure and fluid velocity along any streamline. Equation 3.13 or Equation 3.14 is called the ***Bernoulli equation***.

Remember that it holds only along individual streamlines, not through the entire flow. In other words, the constant in the Equation 3.14 is generally different for each streamline in the flow. And it holds only for inviscid flow, because if the fluid is viscous there are shearing forces across the lateral surfaces of stream tubes, and Newton's second law cannot be written and manipulated so simply. But often in flow of a real fluid the viscous forces are small enough outside the boundary layer that the Bernoulli equation is a good approximation.

27 Note that the right-hand side of Equation 3.13 is the negative of the increase in kinetic energy per unit volume of fluid between point 1 and point 2. The Bernoulli equation is just a statement of the work–energy theorem, whereby the work done by a force acting on a body is equal to the change in kinetic energy of the body. In this case, fluid pressure is the only force acting on the fluid.

28 In discussing inviscid flow around a sphere I called the front and rear points of the sphere the stagnation points, because velocities relative to the sphere are zero there. Using the Bernoulli equation it is easy to find the corresponding *stagnation pressures*. Taking the free-stream values of pressure and velocity to be p_o and v_o, writing Equation 3.13 in the form

$$p - p_o = - \frac{\rho}{2}(v^2 - v_o^2) \tag{3.15}$$

and substituting $v = 0$ at the stagnation points, the stagnation pressures (the same for front and rear points) are

$$p = p_o + \frac{\rho v_o^2}{2} \tag{3.16}$$

TURBULENCE

Introduction

29 Most of the fluid flows of interest in science, technology, and everyday life are turbulent flows—although there are many important exceptions to that generalization, like the flow of groundwater in the porous subsurface, or the flow of blood in capillaries, or the flow of lubricating fluid in thin clearances between moving parts of a machine, or the flow of that thin, slow-moving sheet of water you see on the paved surface of the shopping-mall parking lot after a rain. Because of the range and complexity of problems in turbulent flow, the approach here will necessarily continue to be selective. The introductory material on the description and origin of turbulence in this section is background for the important topic of turbulent flow in boundary layers in the following section and in Chapter 4. The emphasis in all this material on turbulence is on the most important physical effects. Mathematics will be held to a minimum, although some is unavoidable in the derivation of useful results on flow resistance and velocity profiles in Chapter 4.

What Is Turbulence?

30 It is not easy to devise a satisfactory definition of turbulence. ***Turbulence*** might be loosely defined as an irregular or random or statistical component of motion that under certain conditions becomes superimposed on the mean or overall motion of a fluid when that fluid flows past a solid surface or past an adjacent stream of the same fluid with different velocity. This definition does not convey very well what turbulence is really like; it is much easier to describe turbulence than to define it.

Describing Turbulence

31 My goal in this section is to present to you as clear a picture as possible of what turbulence is like. Suppose that you were in possession of a magical instrument that allowed you to make an exact and continuous measurement of the fluid velocity at any point in a turbulent flow as a function of time. I am calling the instrument magical because all of the many available methods of measuring fluid velocity at a point, some of them fairly satisfactory, inevitably suffer to some extent from one or both of two drawbacks: (1) the presence of the instrument distorts or alters the flow one is trying to measure; (2) the effective measurement volume is not small enough to be regarded as a "point".

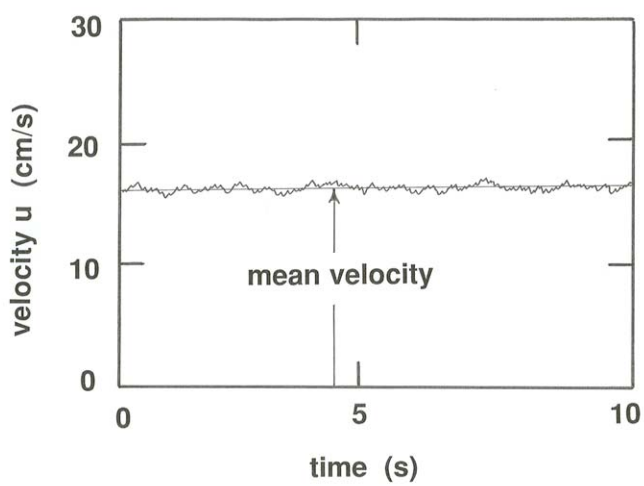

Figure 3-13. Typical record of streamwise instantaneous flow velocity measured at a point in a turbulent channel flow.

32 What would your record of velocities look like? Figure 3-13 is an example of a record, for the component *u* of velocity in the downstream direction. The outstanding characteristic of the velocity is its *uncertainty*: there is no way of predicting at a given time what the velocity at some future time will be. But note that there is a readily discernible (although not precisely definable) range into which most of the velocity fluctuations fall, and the same can be said about the time scales of the fluctuations.

33 Turbulence measurements present a rich field for statistical treatment. First of all, a mean velocity $\overline{u}$ can be defined from the record of u by use of an averaging time interval that is very long with respect to the time scale of the fluctuations but not so long that the overall level of the velocity drifts upward or downward during the averaging time. A fluctuating velocity u' can then be defined as the difference between the instantaneous velocity u and the mean velocity $\overline{u}$:

$$u' = u - \overline{u} \tag{3.17}$$

where the overbar denotes a time average. The time-average value of u' must be zero (by definition!). Now look at the component of velocity in any direction normal to the mean flow direction. You would see a record similar to that shown in Figure 3-13, except that the average value would always have to be zero; the normal-to-boundary velocity is usually called v, and the cross-stream velocity parallel to the boundary and normal to flow) is usually called w. Equations just like Equation 3.17 can be written for the components v and w:

$$\begin{aligned} v' &= v - \overline{v} = v \\ w' &= w - \overline{w} = w \end{aligned} \tag{3.18}$$

34 A good measure of the *intensity* of the turbulence is the root-mean-square value of the fluctuating components of velocity:

$$\begin{aligned} \mathrm{rms}(u') &= \left(u'^2\right)^{1/2} \\ \mathrm{rms}(v') &= \left(v'^2\right)^{1/2} \\ \mathrm{rms}(w') &= \left(w'^2\right)^{1/2} \end{aligned} \tag{3.19}$$

These are formed by taking the square root of the time average of the squares of the fluctuating velocities; for those who are familiar with statistical terms, the rms values are simply standard deviations of instantaneous velocities. They are always positive quantities, and their magnitudes are a measure of the strength or intensity of the turbulence, or the spread of instantaneous velocities around the mean. Turbulence intensities are typically something like five to ten percent of the mean velocity $\overline{u}$ (that is, again in the parlance of statistics, the coefficient of variation of velocity is 5–10%).

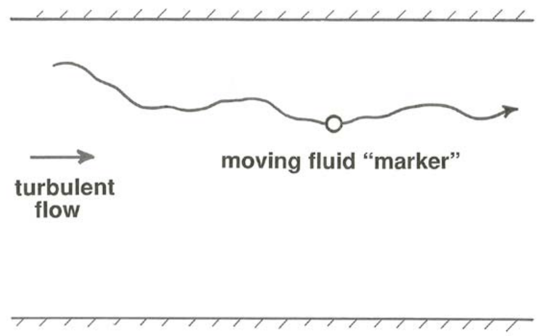

Figure 3-14. Typical trajectory of a small fluid element or neutrally buoyant marker particle in a turbulent channel flow.

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35 Statistical analysis of turbulence can be carried much further than this. But now suppose that you measured velocity in a different way, by following the trajectories of fluid “points” or markers as they travel with the flow and measuring the velocity components as a function of time (Figure 3-14). It is straightforward, though laborious, to do this sort of thing by photographing tiny neutrally buoyant marker particles that represent the motion of the fluid well and then measuring their travel and computing velocities. Velocities measured in this way, called *Lagrangian* velocities, are related to those measured at a fixed point, called *Eulerian* velocities, and the records would look generally similar. The trajectories themselves would be three-dimensionally sinuous and highly irregular, as shown schematically in Figure 3-14, although angles between tangents to trajectories and the mean flow direction are usually not very large, because u' is usually small relative to $\overline{u}$.

36 You can also imagine releasing fluid markers at some fixed point in the flow and watching a succession of trajectories traced out at different times (Figure 3-15). Each trajectory would be different in detail, but they would show similar features.

37 One thing you can do to learn something about the spatial scale of the fluctuations revealed by velocity records like the one in Figure 3-13 is to think about the distance over which the velocity becomes “different” or uncorrelated with distance away from a given point (Figure 3-16). Suppose that you measured the velocity component u simultaneously at two points 1 and 2 a distance x apart in the flow and computed the correlation coefficient by forming products of a large number of pairs of velocities, each measured at the same time, taking the average of all the products, and then normalizing by dividing by the rms value. If the two points are close together compared to the characteristic spatial scale of the turbulence, the velocities at the two points are nearly the same, and the coefficient is nearly one. But if the points are far apart the velocities are uncorrelated (that is, they have no tendency to be similar), and the coefficient is zero or nearly so. The distance over which the coefficient falls to its minimum value, a bit less than one, before rising again to zero is roughly representative of the spatial scale of the turbulent velocity fluctuations. The gentle minimum is an indication that the distance out from the original point corresponds to adjacent eddies, which tend to have an opposing velocity, hence the minimum. A similar correlation coefficient can be computed for Lagrangian velocities, and correlation coefficients can also be based on time rather than on space.

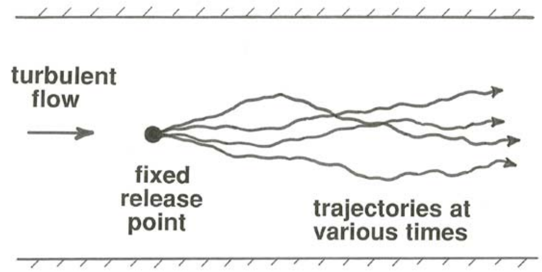

Figure 3-15. A series of trajectories of small, neutrally buoyant markers in a turbulent channel flow, all released from the same point.

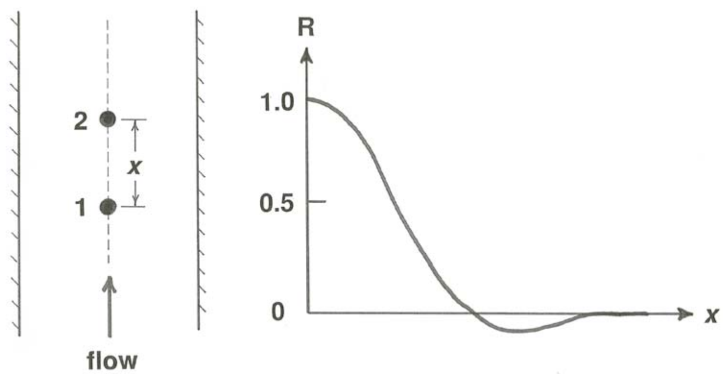

Figure 3-16. Correlation coefficient R for fluid velocity measured at two points, 1 and 2, separated by a streamwise (i.e., alongstream) distance x.

38 One of the very best ways to get a qualitative idea of the physical nature of turbulent motions is to put some very fine flaky reflective material into suspension in a well illuminated flow. The flakes tend to be brought into parallelism with local shearing planes, and variations in reflected light from place to place in the flow give a fairly good picture of the turbulence. Although it is easier to appreciate than to describe the pattern that results, the general picture is one of intergrading swirls of fluid, with highly irregular shapes and with a very wide range of sizes, that are in a constant state of development and decay. These swirls are called turbulent ***eddies***. Even though they are not sharply delineated, they have a real physical existence.

39 The swirly nature of the eddies is most readily perceived when the eye attempts to follow points moving along with the flow; if instead the eye attempts to fix upon a point in the flow that is stationary with respect to the boundaries, fluid elements (if there are some small marker particles contained in the fluid to reveal them) are seen to pass by with only slightly varying velocities and directions, in accordance with the Eulerian description of turbulent velocity at a point.

40 Each eddy has a certain sense and intensity of rotation that tends to distinguish it, at least momentarily, from surrounding fluid. The property of solid-body-like *rotation of fluid at a given point* in the flow is termed ***vorticity***. Think in terms of the rotation of a small element of fluid as the volume of the element shrinks toward zero around the point. The vorticity varies smoothly in both magnitude and orientation from point to point. The eddy structure of the turbulence can be described by how the vorticity varies throughout the flow; the vorticity in an eddy varies from point to point, but it tends to be more nearly the same there than in neighboring eddies.

Laminar and Turbulent Flow

41 At first thought it seems natural that fluids would show a smooth and regular pattern of movement, without all the irregularity of turbulence. Such regular flows are called ***laminar flows***. You will see over and over again in these notes that flows in a given setting or system are laminar under some conditions and turbulent under other conditions. Now that you have some idea of the kinematics of turbulent flow, you might consider what it is that governs whether a given flow is laminar or turbulent in the first place, and what the transition from laminar to turbulent flow is like. Osborne Reynolds did the pioneering work on these questions in the 1880s in an experimental study of flow through tubes with circular cross section (Reynolds, 1883).

42 Think first about the variables that must be important in steady flow through a straight circular tube (Figure 3-17). Density ρ must be taken into account, because of the possibility of turbulent flow in the tube and therefore local fluid accelerations. Viscosity μ must be taken into account because it affects the shearing forces within the fluid and at the wall. A variable that describes the speed of movement of the fluid is important, because this governs both fluid inertia and rates of shear. A good variable of this kind is the mean velocity of flow U in

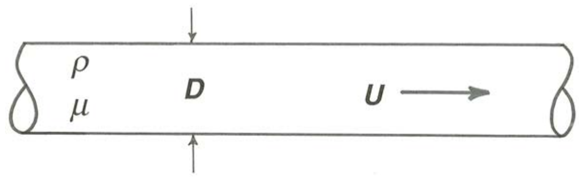

Figure 3-17. Variables associated with steady flow through a circular tube.

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the tube; this can be found either by averaging the local fluid velocity over the cross section of the tube or by dividing the discharge (the volume rate of flow) by the cross-sectional area of the tube. The diameter D of the tube is important because it affects both the shear rate and the scale of the turbulence. Gravity need not be considered explicitly in this kind of flow because no deformable free surface is involved. By dimensional analysis, as discussed in Chapter 1, the four variables U, D, ρ, and μ can be combined into a single dimensionless variable $\rho UD/\mu$ on which all of the characteristics of the flow, including the transition from laminar to turbulent flow, depend. Reynolds first deduced the importance of this variable, now called the ***Reynolds number***, by considering the dimensional structure of the equation of motion in the way I alluded to briefly at the end of Chapter 2.

43 Reynolds made two kinds of experiments. The first, to study the development of turbulent flow from an originally laminar flow, was made in an apparatus like that shown in Figure 3-18: a long tube leading from a reservoir of still water by way of a trumpet-shaped entrance section, through which a flow with varying mean velocity could be passed with a minimum of disturbance. Three different tube diameters (1/4", 1/2", and 1"; 0.64, 1.27, and 2.54 cm) and water of two different temperatures, and therefore of two different viscosities, were used. For each combination of D and μ, U was gradually increased until the originally laminar flow became turbulent. The transition was observed with the aid of a streak of colored water introduced at the entrance of the tube.

> When the velocities were sufficiently low, the streak of color extended in a beautiful straight line through the tube [Figure 3-19A].... As the velocity was increased by small stages, at some point in the tube, always at a considerable distance from the trumpet or entrance, the color band would all at once mix up the surrounding water and fill the rest of the tube with a mass of colored water [Figure 3-19B].... On viewing the tube by the light of an electric spark, the mass of water resolved itself into a mass of more or less distinct curls, showing eddies [Figure 3-19C]

(Reynolds, 1883, p. 942). Reynolds found that for each combination of D and U the point of transition was characterized by almost exactly the same value of Re, around 12,000. Subsequent experiments have since confirmed this over a much wider range of U, D, ρ, and μ.

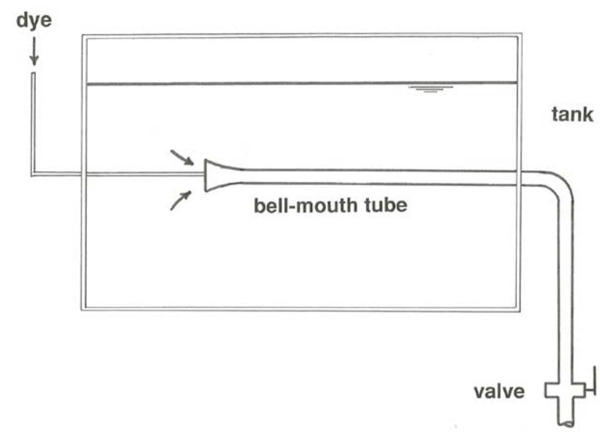

Figure 3-18. Apparatus (schematic) used by Osborne Reynolds in his study of the transition from laminar to turbulent flow in a circular tube.

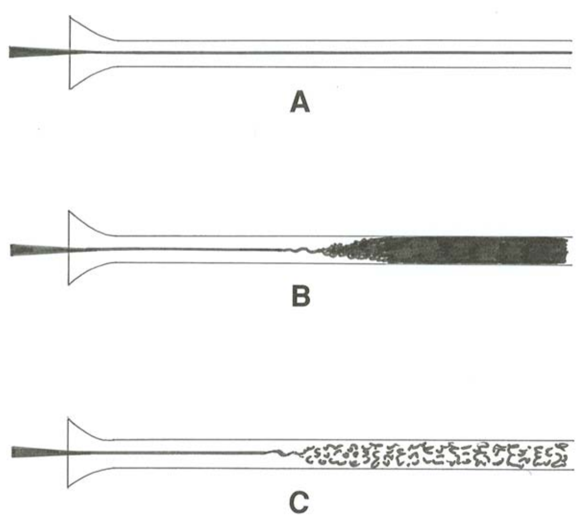

Figure 3-19. The results of Reynolds' experiments on the transition from laminar to turbulent flow in a circular tube.

44 Reynolds suspected that, because the transition from laminar to turbulent flow was so abrupt and the resulting turbulence was so well developed, the laminar flow became potentially unstable to large disturbances at a much lower value of Re than he found for the transition when he minimized external disturbances, and in fact he observed that the transition took place at much lower values of Re if there was residual turbulence in the supply tank or if the apparatus was disturbed in any way. Similar experiments made with even greater care in eliminating such disturbances have since shown that laminar flow can be maintained to much higher values of Re, up to about 40,000, than in Reynolds' original experiments.

45 To circumvent the persistence of laminar flow into the range of Re for which it is unstable, Reynolds made a separate set of experiments to study the transition of originally turbulent flow to laminar flow as the mean velocity in the tube was gradually decreased. To do this he passed turbulent flow through a very long metal pipe and gradually decreased the mean velocity until at some point along the pipe the flow became laminar. The occurrence of the transition was detected by measuring the drop in fluid pressure between two stations about two meters apart near the downstream end of the pipe. (It had been known long before Reynolds' work—and you yourselves will soon be seeing why—that in laminar flow through a horizontal pipe the rate at which fluid pressure drops along the pipe is directly proportional to the mean velocity, whereas in turbulent flow it is approximately proportional to the square of the mean velocity. Thus, although Reynolds could not see the transition he had a sensitive means of detecting its occurrence.) Again many different combinations of D and μ were used, and in every case the transition from turbulent to laminar flow occurred at values of Re close to 2000. This is the value for which laminar flow can be said to be *unconditionally stable*, in the sense that no matter how great a disturbance is introduced, the flow always reverts to being laminar.

Origin of Turbulence

46 Mathematical theory for the origin of turbulence is intricate, and only partly successful in accounting for the transition to turbulent flow at a certain critical Reynolds number. One of the most successful approaches involves analysis of the stability of a laminar flow against very small-amplitude disturbances. The mathematical technique involves introducing a small wavelike disturbance of a certain frequency into the equation of motion for the flow and then seeing whether the disturbance grows in amplitude or is damped. The assumption is that if the disturbance tends to grow it will eventually lead to development of turbulence.

47 Although a satisfactory explanation would take us off the track at this point, in laminar flow there is a tendency for a wave-shaped distortion like the one in Figure 3-20 to be amplified with time: applying the Bernoulli equation along the streamlines shows that fluid pressure is lowest where the velocity is greatest in the region of crowded flow lines, and highest where the velocity is smallest in the region of uncrowded flow lines, and the resulting unbalanced pressure force tends to accelerate the fluid in the direction of convexity and thereby accentuate the distortion. But at the same time the viscous resistance to shearing tends to weaken the shearing in the high-shear part of the distortion and thus tends to make the flow revert to uniform shear. It should therefore seem natural that the Reynolds number, which is a measure of the relative importance of viscous shear forces and accelerational tendencies, should indicate whether disturbances like this are amplified or damped.

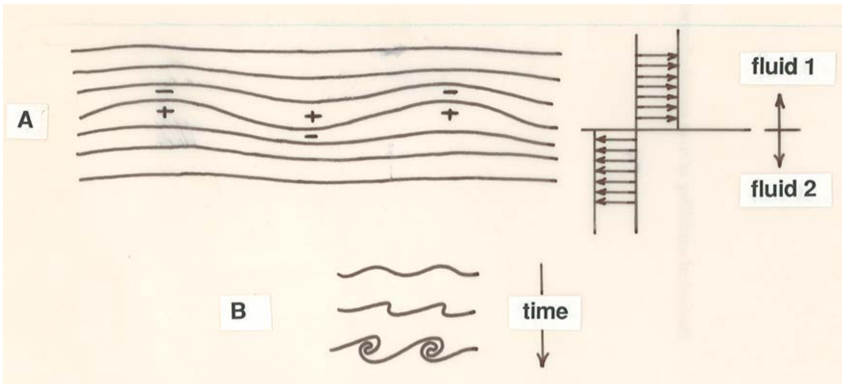

Figure 3-20. Amplification of a wave-shaped disturbance on an interface of velocity discontinuity in laminar flow (schematic). **A)** Pressure forces acting to deform the surface. Plus and minus signs indicate high and low pressures, respectively. **B)** Evolution of the disturbance with time in a series of vortices.

48 Figure 3-21 is a stability diagram for a laminar shear layer or boundary layer (see next section) developed next to a planar boundary. The diagram shows the results of both the mathematical stability analysis described above and experimental observations on stability. The experiments were made by causing a small metal band to vibrate next to the planar boundary at a known frequency and observing the resulting velocity fluctuations in the fluid. Agreement between theory and experiment is good but not perfect; if the experimental results were completely in agreement with the calculated curve, they would all fall on it. The diagram shows that there is a well-defined critical Reynolds number, Re_{crit}, below which the laminar flow is always stable but above which there is a range of frequencies at any Reynolds number for which the disturbance is amplified, so that the laminar flow is potentially unstable and becomes turbulent provided that disturbances with frequencies in that range are present.

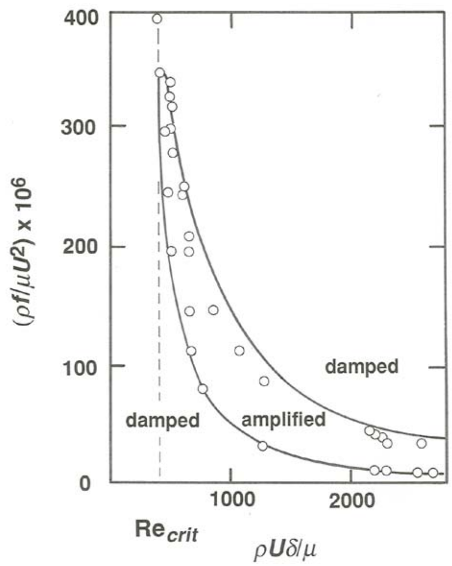

Figure 3-21. Diagram showing stability of a laminar shear layer (boundary layer) developed next to a planar boundary. The vertical axis is a dimensionless measure of the frequency f of the imposed small-amplitude disturbances. The horizontal axis is a Reynolds number based on the thickness δ of the boundary layer and free-stream velocity at the outer edge of the boundary layer. The solid curve is the calculated curve for neutral stability (Lin, 1955); the points represent experimental determinations of neutral stability (Schubauer and Skramstad, 1947).

BOUNDARY LAYERS

Introduction

49 A ***boundary layer*** is the zone of flow in the immediate vicinity of a solid surface or boundary in which the motion of the fluid is affected by the frictional resistance exerted by the boundary. The no-slip condition requires that the velocity of fluid in direct contact with solid boundary be exactly the same as the velocity of the boundary; the boundary layer is the region of fluid next to the boundary across which the velocity of the fluid grades from that of the boundary to that of the unaffected part of the flow (often called the ***free stream***) some distance away from the boundary.

50 Probably the simplest example of a boundary layer is the one that develops on both surfaces of a stationary flat plate held parallel to a uniform free stream of fluid (Figure 3-22). Just downstream of the leading edge of the plate the boundary layer is very thin, and the shearing necessitated by the transition from zero velocity to free-stream velocity is compressed into a thin zone of strong shear, so the shear stress at the surface of the plate is large (cf. Equation 1.8).

Farther along the plate the boundary layer is thicker, because the motion of a greater thickness of fluid is retarded by the frictional influence of the plate, in the form of shear stresses exerted from layer to layer in the fluid; the shearing is therefore weaker, and the shear stress at the surface of the plate is smaller.

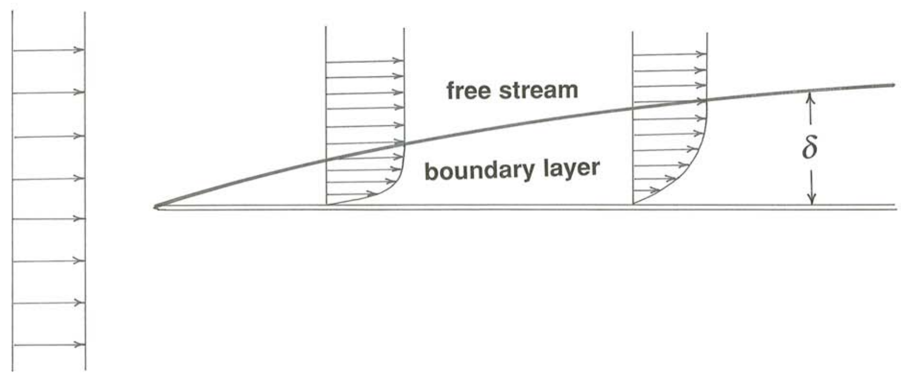

Figure 3-22. Development of a laminar boundary layer on a flat plate at zero incidence (i.e., held edgewise to the flow). A boundary layer develops on both sides of the plate; only one side is shown.

51 Boundary layers develop on objects of any shape immersed in a fluid moving relative to the object: flat plates as discussed above, airplane wings and other streamlined shapes, and blunt or bluff bodies like spheres or cylinders or sediment particles. Boundary layers also develop next to the external boundaries of a flow: the walls of pipes and ducts, the beds and bottoms of channels, the ocean bottom, and the land surface under the moving atmosphere. In every case the boundary layer has to start somewhere, as at the front surface or leading edge of a body immersed in the flow or at the upstream end of any solid boundary to the flow. And in every case it grows or expands downstream, until the flow passes by the body (the shearing motion engendered in the boundary layer is then degraded by viscous forces), or until it meets another boundary layer growing from some other surface, or until it reaches a free surface, or until it is prevented from further thickening by encountering a stably density-stratified layer of the medium—as is commonly the case in the atmosphere and in the deep ocean.

Laminar Boundary Layers and Turbulent Boundary Layers

52 Flow in boundary layers may be either laminar or turbulent. A boundary layer that develops from the leading part of an object immersed in a free stream or at the head of a channel or conduit typically starts out as a laminar flow, but if it has a chance to grow for a long enough distance along the boundary it abruptly becomes turbulent. In the example of a flat-plate boundary layer (Figure 3-23) we can define a Reynolds number $Re_\delta = \rho U\delta/\mu$ based on free-stream velocity U and boundary-layer thickness δ; just as in flow in a tube, discussed in a previous section, past a certain critical value of Re_δ the laminar boundary layer is potentially unstable and may become turbulent. If there are no large turbulent eddies in the free stream, the laminar boundary layer may persist to very high Reynolds numbers; if the free stream is itself turbulent, or if the solid boundary surface is very rough, the boundary layer may become turbulent a very short distance downstream of the leading edge. Turbulence in the form of small spots develops at certain points in the laminar boundary layer, spreads rapidly, and soon engulfs the entire boundary layer.

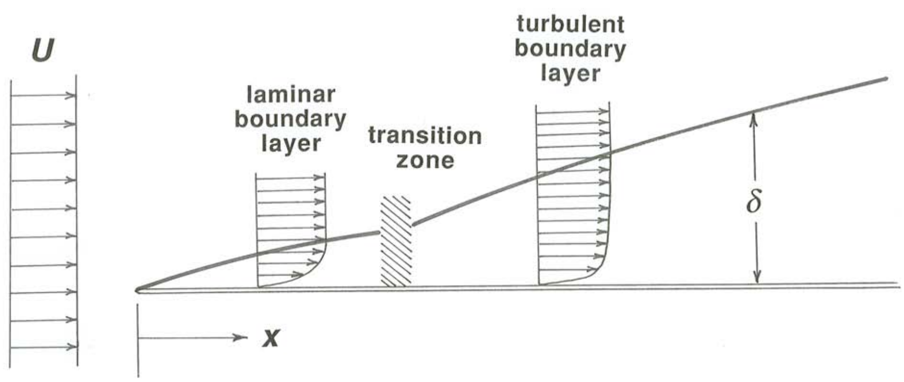

Figure 3-23. Transition from a laminar boundary layer to a turbulent boundary layer on a flat plate at zero incidence.

53 Once the boundary layer becomes turbulent it thickens faster, because fluid from the free stream is incorporated into the boundary layer at its outer edge in much the same way that clear air is incorporated into a turbulent plume of smoke (Figure 3-24). That effect is in addition to, and as important as, the effect of incorporation of new fluid into the boundary layer just by local frictional action—which is the *only* way a laminar boundary layer can thicken. But the thickness of even a turbulent boundary layer grows fairly slowly relative to downstream distance; the angle between the average position of the outer edge of the boundary layer and the boundary itself is not very large, typically something like a few degrees.

Wakes

54 In situations where the flow passes all the way past some object of finite size surrounded by the flow, the boundary layer does not have a chance to develop beyond the vicinity of the body itself (Figure 3-25). Downstream of the object the fluid that was retarded in the

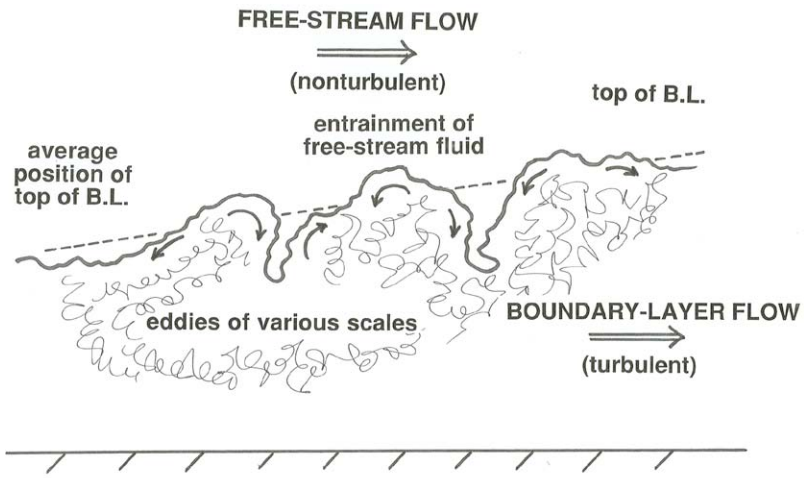

Figure 3-24. Sketch of processes acting to thicken a turbulent boundary layer on a flat plate at zero incidence.

boundary layer is gradually reaccelerated by the free stream, until far downstream the velocity profile in the free stream no longer shows any evidence of the presence of the object upstream. The zone of retarded and often turbulent fluid downstream of the object is called the ***wake***.

How Thick are Boundary Layers?

55 One usually thinks of a boundary layer as being thin compared to the scale of the body on which it develops. This is true at high Reynolds numbers, but it is not true at low Reynolds numbers. I will show you here, by a fairly simple line of reasoning, that the boundary-layer thickness varies inversely with the Reynolds number.

56 The thickness of the boundary layer is determined by the relative magnitude of two effects: (1) the slowing of fluid farther and farther away from the solid surface by the action of fluid friction, and (2) the sweeping of that low-momentum fluid downstream and its replacement by fluid from upstream moving at the free-stream velocity. The greater the second effect compared with the first, the thinner the boundary layer.

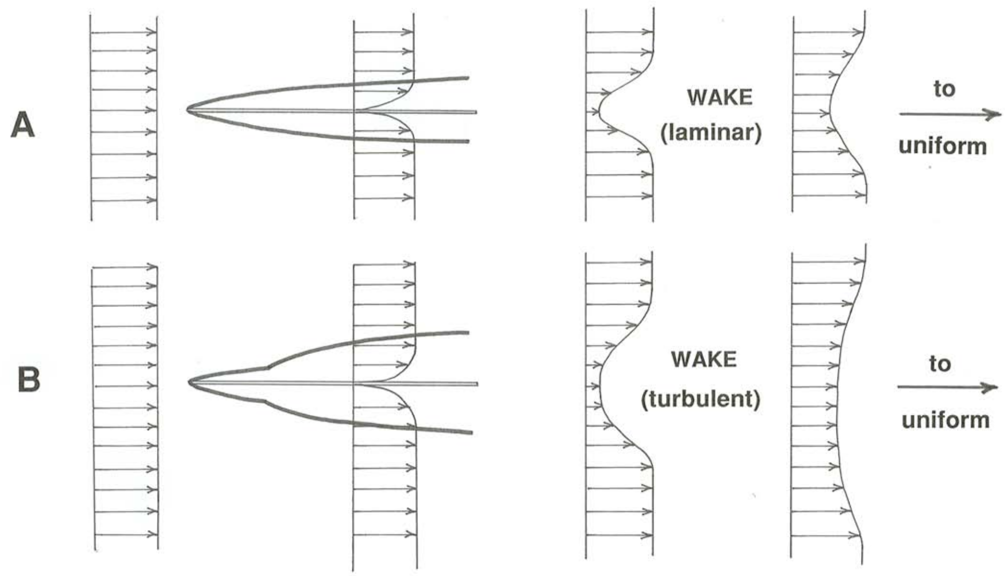

Figure 3-25. Development of a wake downstream of a flat plate at zero incidence.

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57 Think in terms of the downstream component of fluid momentum at some distance away from the solid boundary and at some distance downstream from the leading edge of the boundary layer. The rate of downstream transport of fluid momentum (written per unit volume of fluid) at the outer edge of the boundary layer is $U(\rho U)$, where U is the free-stream velocity. The slowing of fluid by friction is a little trickier to deal with. Think back to Chapter 1, where I introduced the idea that the viscosity can be thought of as a cross-stream diffusion coefficient for downstream fluid momentum. In line with that idea, within the boundary layer the downstream fluid momentum is all the time diffusing toward the boundary. (Fluid dynamicists like to say that the boundary is a *sink* for momentum.) So the rate of cross-stream momentum diffusion is approximately equal to $\mu(U/\delta)$, where U/δ represents in a crude way the velocity gradient du/dy within the boundary layer.

58 The rate of thickening of the boundary layer is crudely represented by the ratio of downstream transport of momentum, on the one hand, to the rate of decrease of momentum at a place on account of the diffusion of momentum toward the boundary, both of these quantities having been derived in the last paragraph:

$$\frac{\text{cross-stream diffusion}}{\text{downstream transport}} = \frac{\mu U/\delta}{\rho U^2}$$

$$= \frac{\mu}{\rho U \delta}$$

$$= 1/Re_{\delta} \qquad (3.20)$$

59 Equation 3.20 shows that the rate of boundary-layer thickening varies as the inverse of the Reynolds number based on boundary-layer thickness. This means that the boundary layer thickens more and more slowly in the downstream direction, so the cartoon of the flat-plate boundary layer in Figure 3-22, with the top of the boundary layer describing a curve that is concave toward the plate, is indeed qualitatively correct.

60 Equation 3.20 also tells you that the larger the Reynolds number based on the mean flow and the size of the solid object on which the boundary layer is growing, the thinner the boundary layer is at a given point—because for given δ, Re_{δ} is proportional to this Reynolds number. (For the flat plate, this Reynolds number is based on the distance from the leading edge; for the sphere, it is based most naturally on sphere diameter.) So the faster the free stream velocity and the larger the sphere (or the farther down the flat plate), and the smaller the viscosity, the thinner the boundary layer.

61 Keep in mind, as a final note, that all of the foregoing is for a laminar boundary layer—although the second part of the conclusion, that boundary-layer thickness is proportional to some Reynolds number defined on the size of the body, is qualitatively true for a turbulent boundary layer as well.

62 You might be wondering how thick boundary layers really are. This is something you can think about the next time you are sitting in a window seat over the wing, several miles above the Earth. How thick is the boundary layer at a distance of, say, one meter from the leading edge of the wing, when the plane is traveling at 500 miles an hour? There is an exact solution for the thickness of a laminar boundary layer as a function of the Reynolds number Re_x based on free-stream velocity and distance from the leading edge:

$$\delta = 4.99\ Re_x^{-1/2} \qquad (3.19)$$

(The derivation of Equation 3.21 is a little beyond this course; see Tritton, 1988, p. 127–129 if you are interested in pursuing it further.) Assuming an air temperature of -50°C and an altitude of 35,000 feet, the density of the air is about 10^{-3} g/cm^3 and the viscosity is something like 1.5 x 10^{-4} poise. Substituting the various values into Equation 3.21, we find that the boundary-layer thickness is a few hundredths of a millimeter. The boundary layer on the roof of your car at 65 mph is much thicker, by about an order of magnitude, because the air speed is so much slower.

Some Flows Are "All Boundary Layer"

63 An example of the boundary layer growing to fill the entire flow is an open-channel flow that has just emerged from a sluice-like outlet at the bottom of a large reservoir of water (Figure 3-26). Right at the inlet, the entire flow could be considered the "free stream". As the flow passes down the channel, a boundary layer grows upward into the flow from the bottom. If the minor effect of friction with the atmosphere is neglected, no boundary layer develops at the upper surface of the flow. Eventually the growing boundary layer reaches the surface, and from that point downstream the river is all boundary layer!

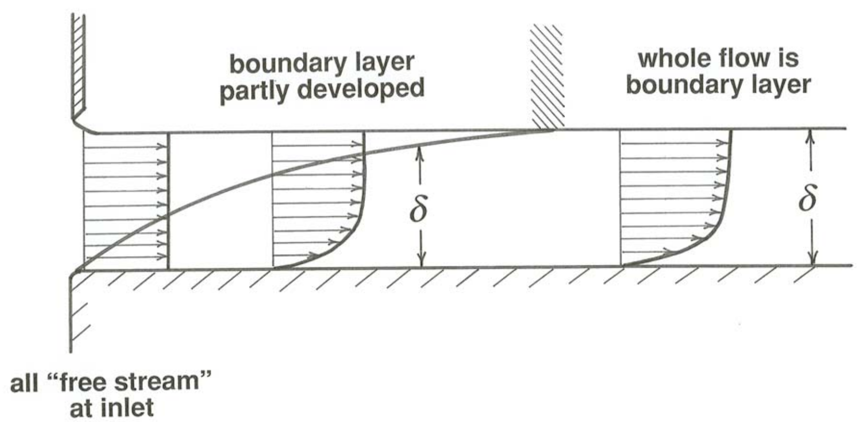

Figure 3-26. Downstream development of a boundary layer in an open-channel flow that begins at the outlet of a sluice gate.

64 In a situation like this, boundary-layer development is typically complete in a downchannel distance equal to something like a few tens of flow depths. Upstream, in the zone of boundary-layer growth, the boundary layer is nonuniform, in that it is different at each section; downstream, in the zone of fully established flow, the boundary layer is uniform, in that it looks the same at every cross section.

Internal Boundary Layers

65 Finally, there can be boundary layers within boundary layers. Such boundary layers are called ***internal boundary layers***. Suppose that a thick boundary layer is developing on a broad surface in contact with a flow, or a boundary layer has already grown to the full lateral extent of the flow, as in a river. Any solid object of restricted size immersed in that boundary layer, located either on the boundary, like some kind of irregularity or protuberance, or within the flow, like part of a submarine structure, causes the local development of another boundary layer (Figure 3-27).

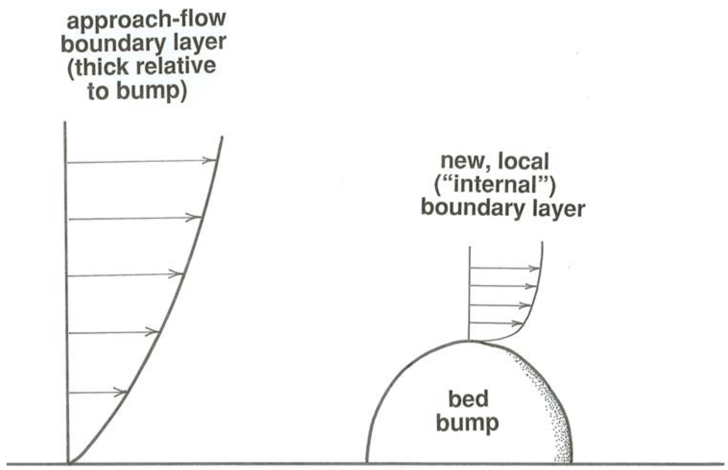

Figure 3-27. Development of an internal boundary layer on a hemispherical roughness element on the bed of a channel flow.

FLOW SEPARATION

66 The overall pattern of flow at fairly high Reynolds numbers past blunt bodies or through sharply expanding channels or conduits is radically different from the pattern expected from inviscid theory, which I have said is often a good guide to the real flow patterns. Figure 3-28 shows two examples of such flow patterns, one for a sphere and one for a duct or pipe that has a downstream expansion at some point. Near the point where the solid boundary begins to diverge or fall away from the direction of the mean flow, the boundary layer separates or breaks away from the boundary. This phenomenon is called ***flow separation***.

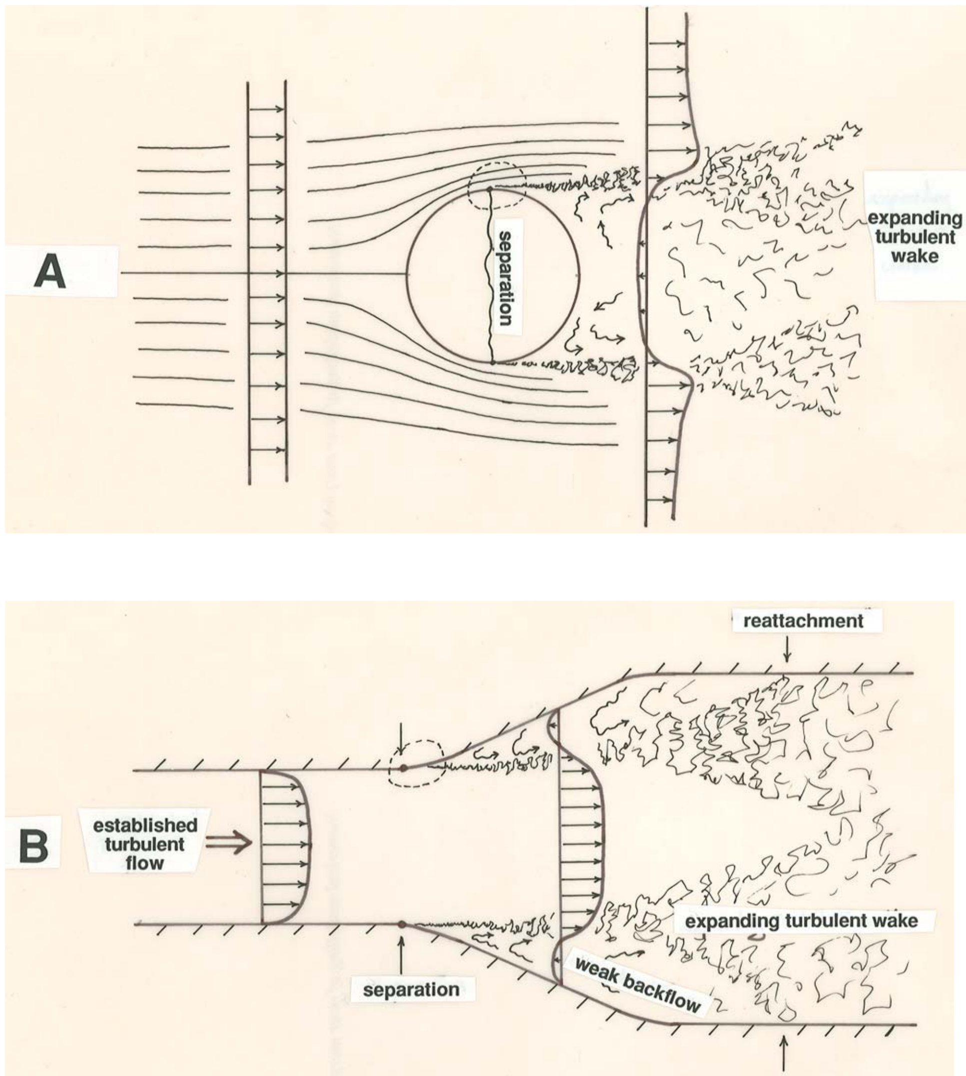

Figure 3-28. Two examples of flow separation: **A)** flow around a sphere; **B)** flow through an expansion in a planar duct.

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67 In all cases the flow separates from the boundary in such a way that the fluid keeps moving straight ahead as the boundary surface falls away from the direction of flow just upstream. The main part of the flow, outside the boundary layer, diverges from the solid boundary correspondingly. If you look only at the regions enclosed by the dashed curves in Figure 3-28 you can appreciate that flow separation is dependent not so much on the overall flow geometry as on the change in the orientation of the boundary relative to the overall flow—a change that involves a curving away of the boundary from the overall flow direction. Separation takes place at or slightly downstream of the beginning of this curving away.

68 The region downstream of the separation point is occupied by stagnant fluid with about the same average velocity as the boundary itself. In this region the fluid has an unsteady eddying pattern of motion, with only a weak circulation as shown in Figure 3-28. As soon as the boundary layer leaves the solid boundary it is in contact with this slower-moving fluid across a surface of strong shear. This surface of shearing is unstable, and a short distance downstream of the separation point it becomes wavy and then breaks down to produce turbulence. This turbulence is then mixed or diffused both into the main flow and into the stagnant region, and it is eventually damped out by viscous shearing within eddies, but its effect extends for a great distance downstream. The stagnant region of fluid inside the separation surface, together with the region of strong turbulence developed on the separation surface, is called a ***wake***. Far downstream from a blunt body like a sphere (Figure 3-28A) the wake turbulence is weak and the average fluid velocity along a profile across the mean flow is slightly less than the free-stream velocity. In flow past an expansion in a duct or channel (Figure 3-28B), the expanding zone of wake turbulence eventually impinges upon the boundary; downstream of this point, where the flow is said to *reattach* to the boundary, the flow near the boundary is once again in the downstream direction, and a new boundary layer develops until far downstream of the expansion the flow is once again fully established.

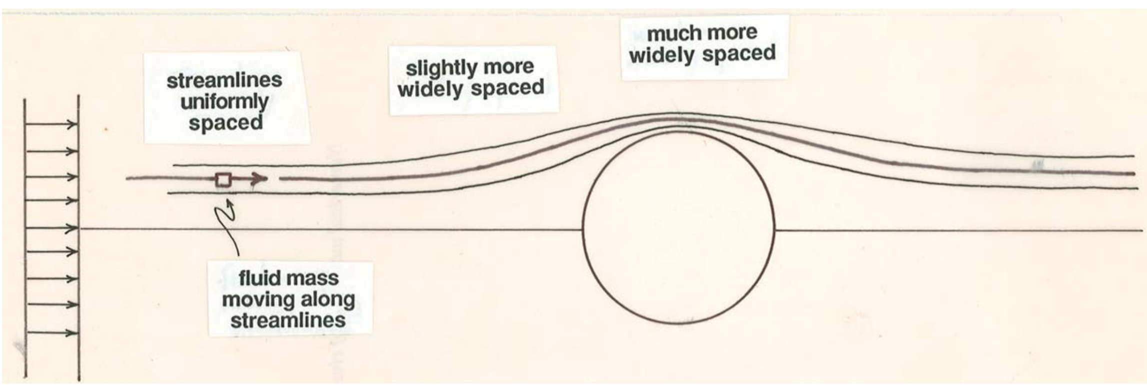

Figure 3-29. Pattern of streamlines in steady inviscid flow past a sphere.

69 You can understand why flow separation takes place by reference to steady inviscid flow around a sphere (Figure 3-29). Remember that variations in fluid velocity can be deduced qualitatively just from variations in spacing of neighboring streamlines. As a small mass of fluid approaches the sphere along a streamline that will take it close to the surface of the sphere, it decelerates slightly from its original uniform velocity and then accelerates to a maximum velocity at the midsection of the sphere (Figure 3-30A). Beyond the midsection it experiences precisely the reverse variation in velocity: it decelerates to minimum velocity and then accelerates slightly back to the free-stream velocity. We can apply the Bernoulli equation (Equation 3.13 or 3.14) to find the corresponding variation in fluid pressure (Figure 3-30B). The pressure is slightly greater than the free-stream value at points just upstream and just downstream of the sphere but shows a minimum at the midsection. It is this variation in pressure that causes strong accelerations and decelerations as the fluid passes around the sphere. In front of the sphere the pressure decreases along the streamline (the spatial rate of change or gradient of pressure is said to be negative or *favorable*), so there is a net force on the fluid mass in the direction of motion, causing an acceleration. In back of the sphere the pressure increases along the streamline (the pressure gradient is positive or *adverse*), so there is a net force opposing the motion, and the fluid mass decelerates.

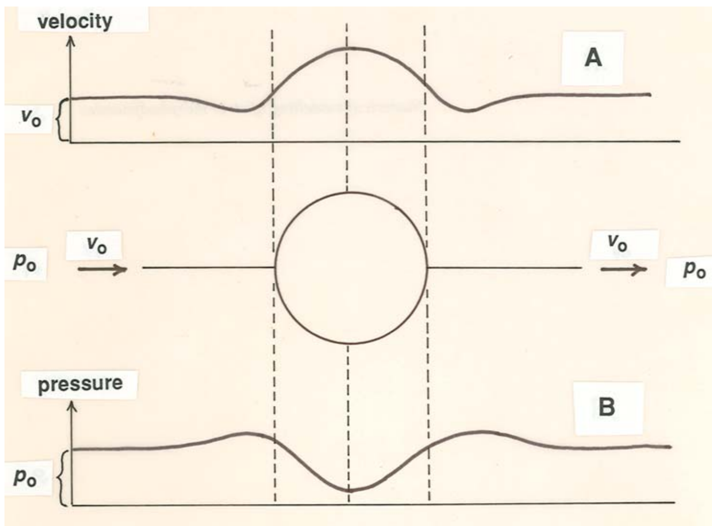

Figure 3-30. Variation in **A)** velocity and **B)** pressure along a streamline passing close to the surface of a sphere, for steady inviscid flow past the sphere (schematic).

70 In inviscid flow the pressure is the only force in the fluid. But in the real world of viscous fluids, a boundary layer develops next to the sphere (Figure 3-31). If the boundary layer is thin, the streamwise variation in fluid pressure given by the Bernoulli equation along streamlines just outside the boundary layer is approximately the same as the pressure on the boundary; the pressure outside the boundary layer is said to be *impressed* on the boundary. If now you follow the motion of a fluid mass along a streamline that is close enough to the sphere to become involved in the boundary layer, a viscous force as well as the impressed pressure force acts on the fluid mass. Because the viscous force everywhere opposes the motion, the fluid mass cannot ultimately regain its uniform velocity after passing the sphere, as in inviscid flow. The fluid cannot accelerate as much in front of the sphere as in the inviscid flow, and it reaches the midsection with lower velocity; then the adverse pressure gradient in back of the sphere, which is augmented by the viscous retardation, decelerates the fluid to zero velocity and causes it to start to move in reverse. This reverse flow forms a barrier to the continuing flow from the front of the sphere, and so the flow must break away from the boundary to pass over the obstructing fluid. Because velocities are small along streamlines close to the boundary, this deceleration to zero velocity occurs only a short distance downstream of the onset of the adverse pressure gradient where the boundary curves away from the mean flow direction.

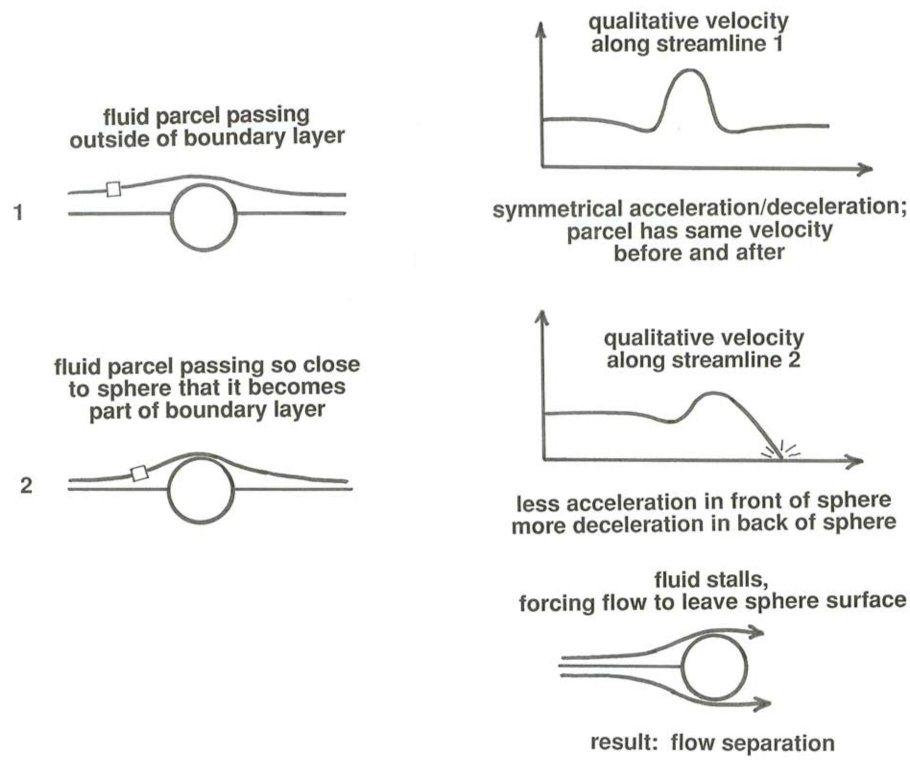

Figure 3-31. Flow processes leading to the onset of flow separation.

71 Once the separated flow is established, the flow pattern looks something like that shown in Figure 3-32. This figure is just a detail of the region enclosed by the dashed curves in Figure 3-28.

72 You might justifiably ask why this same explanation should not hold just as well for slow flow around a sphere at Reynolds numbers small enough to be in the Stokes range. A superficial answer would be that according to Stokes' law for slow viscous flow around a sphere the distributions of pressure and shear stress are such that the flow passes around the sphere without reversal. A more basic explanation, which is qualitatively true but may not be very helpful, is as follows. As noted earlier in this chapter, flow around a sphere at low velocities is characterized by fluid accelerations that are everywhere so small compared to fluid velocities that the viscous forces are everywhere closely balanced by pressure forces, so that there is no tendency for fluid to decelerate to a stall. At these low velocities, retardation by viscous shearing in the fluid caused by the presence of the solid boundary extends for a great distance away from the surface of the sphere. As the velocity around the sphere increases, this retarded fluid is to a progressively greater extent swept or advected back around the sphere, to be "replaced" by faster-moving fluid, thus concentrating the region of retardation into a relatively thin layer near the solid boundary. The pressure distribution in the fluid outside this thin boundary layer becomes more and more like that predicted by inviscid theory. Think in terms of a balance between spreading of retardation outward from the solid boundary, on the one hand, and delivery of faster-moving fluid from upstream, on the other hand. As the Reynolds number increases, the latter effect becomes more and more important relative to the former. Ultimately, flow separation develops for the reasons outlined above.

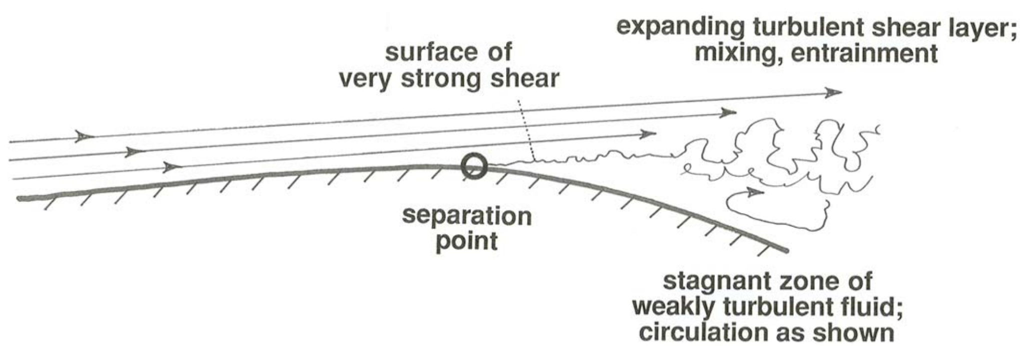

Figure 3-32. Close-up view of flow separation (schematic).

FLOW PAST A SPHERE AT HIGH REYNOLDS NUMBERS

73 So far we have considered flow past a sphere only from the standpoint of dimensional analysis, in Chapter 2, to derive a relationship between drag coefficient and Reynolds number, and we have looked at flow patterns and fluid forces only at very low Reynolds numbers, in the Stokes range. You are now equipped to deal with flow past a sphere at higher Reynolds numbers.

74 As the Reynolds number increases, flow separation gradually develops, and this corresponds to a change from a regime of flow dominated by viscous effects, with viscous forces and pressure forces about equally important, to a regime of flow dominated by flow-separation effects, with pressure forces far larger than viscous forces. This gradual change in the flow regime is manifested in the change from the descending-straight-line branch of the curve for drag coefficient C_D as a function of Reynolds number (see Figure 2-2) to the approximately horizontal part of the curve at higher Reynolds numbers. Even before separation is fully developed, there are deviations of the observed drag coefficient from that predicted by Stokes' law (Figure 3-33), but, after flow separation well established, the curve for C_D shows no relationship whatsoever to Stokes' law (Figure 2-2).

75 In this section we will examine in a qualitative way the gradual but fundamental ways the flow pattern around the sphere changes as the Reynolds number increases. These changes can be classified or subdivided into several stages, which could well be called flow regimes. ***Flow regimes*** are distinctive or characteristic patterns of

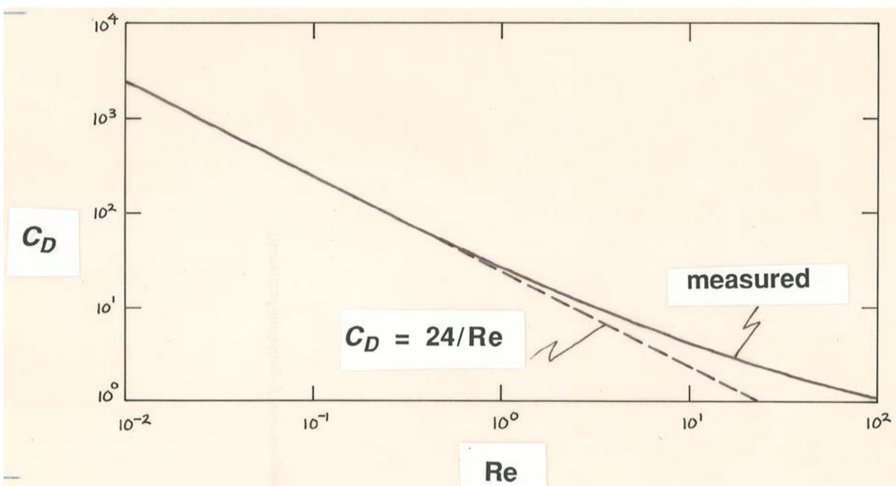

Figure 3-33. Deviation of drag coefficient C_D from Stokes' law at Reynolds numbers between 1 and 100.

flow, which are manifested in certain definite ranges of flow conditions and which are qualitatively different from other regimes that are manifested in neighboring ranges of flow conditions. The flow regimes associated with flow around a sphere are intergradational but distinctive. Keep in mind that they are characterized or described completely by the Reynolds number, and only by the Reynolds number: it is not just the size of the sphere, or the velocity of flow around it, or the kind of fluid; it is how all of these combine to give a particular value of the Reynolds number.

76 Figure 3-34 shows a cartoon series of flow patterns with increasing Re, and the corresponding position on the drag-coefficient curve (Figure 2-2). Looking ahead to the following section on settling of spheres, these figures also give approximate values of the diameters of quartz spheres settling in water at the given Reynolds numbers, and the corresponding settling velocity, in centimeters per second.

77 Figure 3-34A shows the picture for creeping flow at Re << 1, as already discussed. The streamlines show a symmetrical pattern front to back. Although not shown in the figure, the flow velocity increases only gradually away from the surface of the sphere; in other words, there is no well-defined boundary layer at these low Reynolds numbers.

78 In Figure 3-34B, for Re ≈ 1, the picture is about the same as at lower Re, but streamlines converge more slowly back of the sphere than they diverge in front of the sphere. Corresponding to this change in flow pattern, it is in about this range that the front-to-back pressure forces begin to increase more rapidly than predicted by Stokes' Law.

79 Flow separation can be said to begin at a Reynolds number of about 24. The point of separation is at first close to the rear of the sphere, and separation results in the formation of a ring eddy attached to the rear surface of the sphere. Flow within the eddy is at first quite regular and predictable (Figure 3-34C), thus not turbulent, but, as Re increases, the point of separation moves to the side of the sphere, and the ring eddy is drawn out in the downstream direction and begins to oscillate and become unstable (Figure 3-34D). At Re values of several hundred, the ring eddy is cyclically shed from behind the sphere to drift downstream and decay as another forms (Figure 3-34E). Also in this range of Re, turbulence begins to develop in the wake of the sphere. At first turbulence develops mainly in the thin zone of strong shearing produced by flow separation and then spreads out downstream, but as Re reaches values of a few thousand the entire wake is filled with a mass of turbulent eddies (Figure 3-34F).

80 In the range of Re from about 1000 to about 200,000 (Figure 3-34F) the pattern of flow does not change much. The flow separates at a position about 80° from the front stagnation point, and there is a fully developed turbulent wake. The drag is due mainly to the pressure distribution on the surface of the sphere, with only a minor contribution from viscous shear stress. The pressure distribution is as shown in Figure 3-35 and does not vary much with Re in this range, so the drag coefficient remains almost constant at about 0.5.

81 At very high Re, above about 200,000, the boundary layer finally becomes turbulent before separation takes place, and there is a sudden change in the flow pattern (Figure 3-34G). The distinction here is between ***laminar separation***, in which the flow in the boundary layer is still laminar where separation takes place, and ***turbulent separation***, in which the boundary layer has already changed from being laminar to being turbulent at some point upstream of separation. Turbulent separation takes place farther around toward the rear of the sphere, at a position about 120–130° from the front stagnation point. The wake becomes contracted compared to its size when the separation is laminar, and

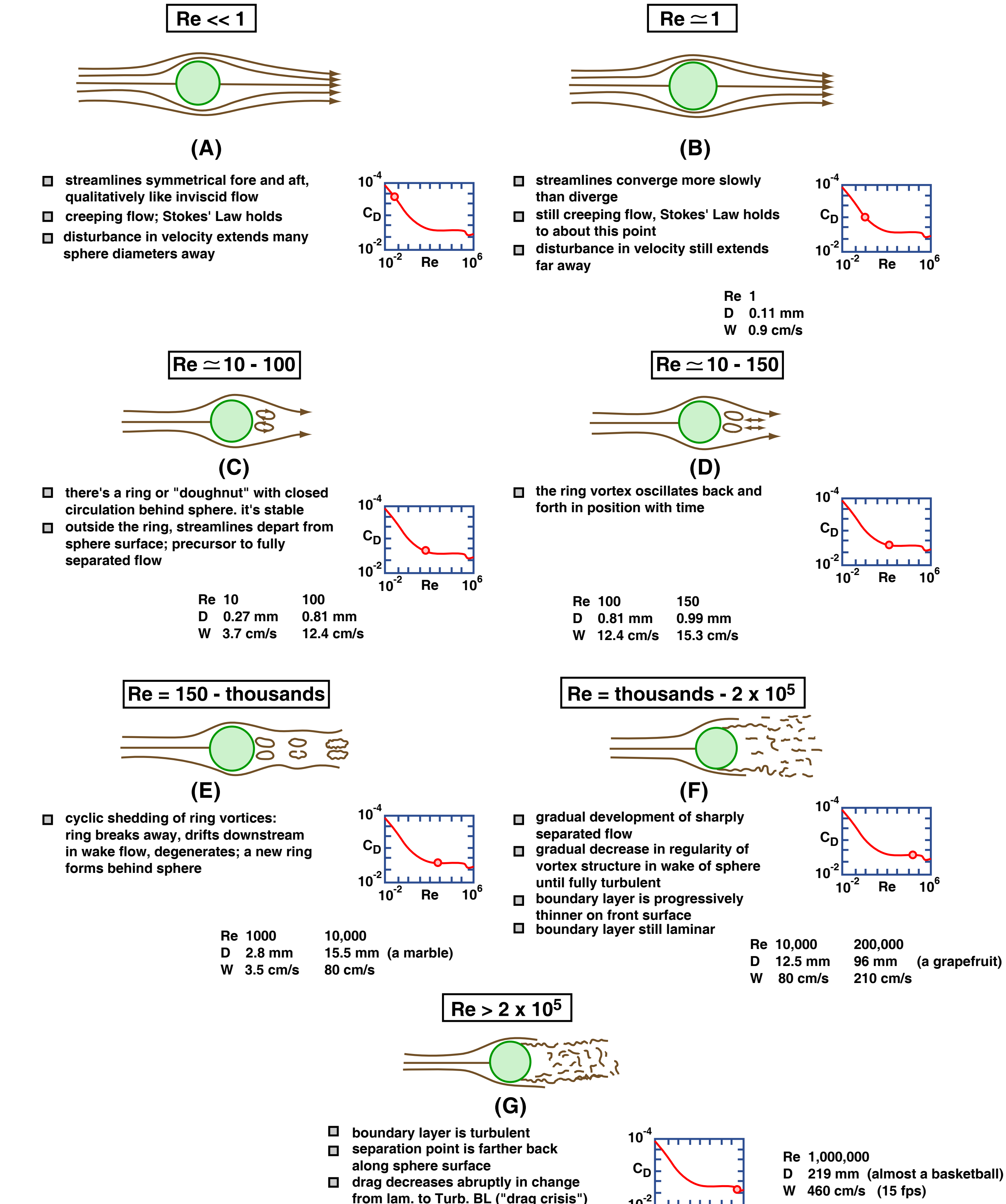

Figure by MIT OpenCourseWare.

schematic patterns of flow around a sphere for several values of Reynolds number $\rho UD/\mu$. for each part, the corresponding location of the curve of drag coefficient C_D vs. Reynolds number (cf. Figure 2-2) is shown, as well as settling velocities and sizes of quartz spheres settling through room-temperature water at various values of Reynolds number.

consequently the very low pressure exerted on the surface of the sphere within the separation region acts over a smaller area. Also, the pressure itself in this region is not as low (Figure 3-35). The combined result of these two effects is a sudden drop in the drag coefficient C_D, to a minimum of about 0.1. This is sometimes called the ***drag crisis***.

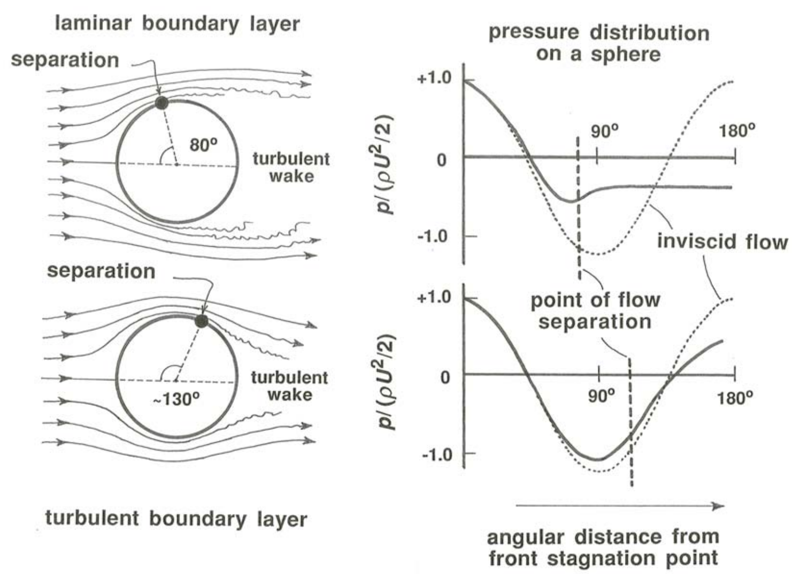

Figure 3-35. Flow patterns and pressure distributions around a sphere at high Reynolds numbers. A) Experimental results for a laminar boundary layer; B) results for a turbulent boundary layer. In each case, the theoretical pressure distribution for inviscid flow is shown for comparison. The pressure is scaled by the stagnation pressure, $\rho U^2/2$.

82 Have you ever wondered why golf balls have that pattern of dimples on them? It is to make them go faster and farther, but why? It is because the Reynolds number of the flying golf ball is just about in the range of transition from a laminar boundary layer to a turbulent boundary layer, and the dimples help to trigger the transition and thus reduce the air drag on the flying ball.

SETTLING OF SPHERES

Introduction

83 This section deals with some basic ideas about settling of solid spheres under their own weight through still fluids. This is an important topic in meteorology (hailstones), sedimentology (sediment grains), and technology (cannon balls and spacecraft). In this section we will look at the terminal settling velocity of spheres as an applied problem. At the end I will make some comments about the complicated matter of the time and distance it takes for a sphere to attain its terminal settling velocity.

84 If placed in suspension in a viscous fluid, a solid body denser than the fluid settles downward and a solid body less dense than the fluid rises upward. A qualification is needed here, however: the body must not be so small that its submerged weight is even smaller than the random forces exerted on it by bombardment by the fluid molecules in thermal motion. Such small weights are generally associated only with the finest particles, in the colloidal size range of small fractions of a micron.

85 When a nonneutrally buoyant body is released from rest in a still fluid, it accelerates in response to the force of gravity. As the velocity of the body increases, the oppositely directed drag force exerted by the fluid grows until it eventually equals the submerged weight of the body, whereupon the body no longer accelerates but falls (or rises) at its ***terminal velocity***, also called the ***fall velocity*** or ***settling velocity*** in the case of settling bodies (Figure 3-36).

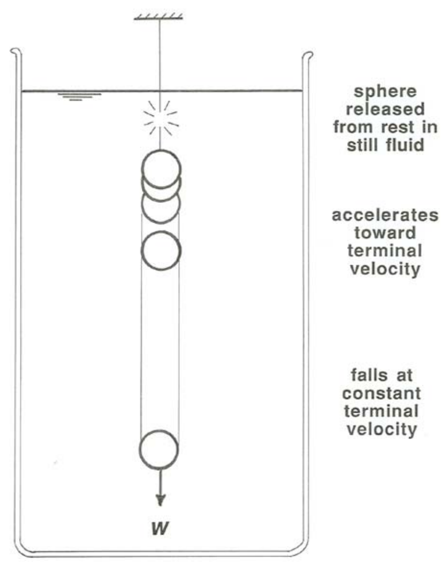

Figure 3-36. Attainment of terminal fall velocity when a sphere is dropped from rest in a vessel of still liquid.

Towing vs. Settling

86 I promised you in Chapter 2 that I would make some comments later about the differences between moving a sphere through still fluid and passing a moving fluid by a stationary sphere. This topic has relevance to the settling of spheres, so I will say some things about it at this point.

87 It should make sense to you that towing a sphere at velocity U through a still fluid by exerting a constant force F_D on it is equivalent to passing a steady and uniform stream of fluid at velocity U around a sphere that is held fixed relative to the boundaries of the flow. This is largely true, but there are two complications. First, if the sphere is held fixed and the flow passes by it, the drag force can be influenced by even weak turbulence in the approaching flow, whereas if the sphere is towed through still fluid there can be no such effect. Second, you have seen that, in some ranges of relative velocity, eddies can form behind the sphere and break away irregularly; if the sphere is fixed and the fluid is flowing by, this causes the force to fluctuate about some average value but does not affect the relative velocity, whereas if the sphere is towed, either the velocity fluctuates along with the force or, if by definition we tow with a constant force, the velocity fluctuates but the force is steady.

88 Settling of a sphere through still fluid under its own weight is exactly like towing the sphere vertically downward by applying a constant towing force, namely the weight of the sphere, which is simply the Earth's gravitational attractive force on the sphere. The weight of the sphere is constant and entirely independent of the state of motion, and the sphere responds by settling downward at some velocity through the fluid. (As noted above, this velocity may fluctuate slightly with time.) For spheres the differences between the fixed-sphere case and the settling-sphere case are usually assumed to be minor. Indeed, some of the data in Figure 2-2 for dimensionless drag force as a function of Reynolds number are from settling experiments and some are from wind-tunnel experiments with fixed spheres, and it can be seen that there is very little scatter of the combined experimental curve.

Dimensional Analysis

89 To obtain an experimental curve for settling velocity we can simply transform the curve in Figure 2-2 for drag coefficient vs. Reynolds number for towed spheres into a curve based on settling velocity. In fact, much of this curve, especially for low Reynolds numbers, was obtained by settling experiments in the first place, with the experimental results recast into the form of drag coefficients.

90 When a sphere falls at terminal velocity the drag force F_D is equal to the submerged weight of the particle, $(1/6)\pi D^3\gamma'$, where γ' is the submerged weight per unit volume of the particle, equal to $g(\rho_s - \rho)$. Substituting this for F_D in the definition of the drag coefficient C_D in Equation 2.3, using settling velocity w in place of U, and then solving for C_D,

$$C_D = \frac{4}{3}\frac{\gamma' D}{\rho w^2} \tag{3.22}$$

This expression for C_D, which can be viewed as the "settling drag coefficient", can be used in the relationship for dimensionless drag force as a function of Reynolds number (Equation 2.3) for spheres moving through a viscous fluid:

$$\frac{\gamma' D}{\rho w^2} = f\left(\frac{\rho w D}{\mu}\right) \tag{3.23}$$

where the factor 4/3 has been absorbed into the function, just for convenience. Figure 3-37, which is the same as Figure 2-2 with axes relabeled and adjusted in scale to take account of the factor 4/3, is the

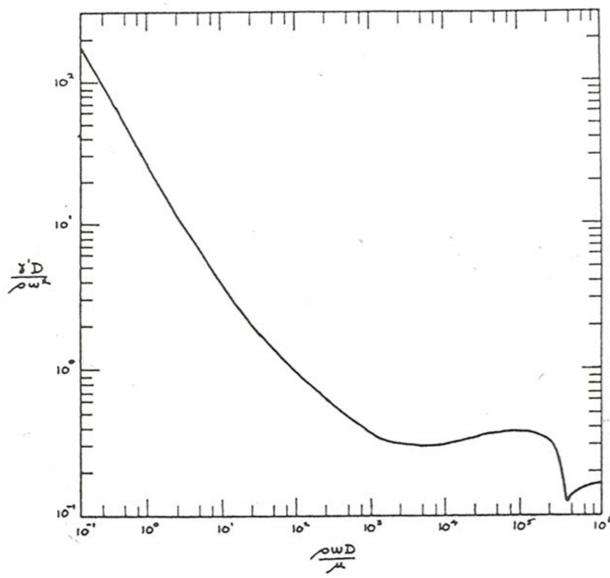

Figure 3-37. "Settling drag coefficient" $\gamma D/\rho w^2$ vs. Reynolds number $\rho w D/\mu$ based on settling velocity.

corresponding graph of this function. No data points are shown, because the curve is exactly the same as in Figure 2-3. Figure 3-37 gives settling velocity w as an implicit function of ρ, U, D, and γ'.

91 The curve in Figure 3-37 is still not very convenient for finding the settling velocity when the other variables are given. This is because both w and D appear in the dimensionless variables along both axes. Finding w in an actual problem would necessitate laborious trial-and-error computation. To get around this problem the graph can be further recast into a more convenient form in which w appears in only one of the two dimensionless variables. Also, because usually what is desired is w as a function of D, or vice versa, it is convenient to arrange for D to appear only in the other variable.

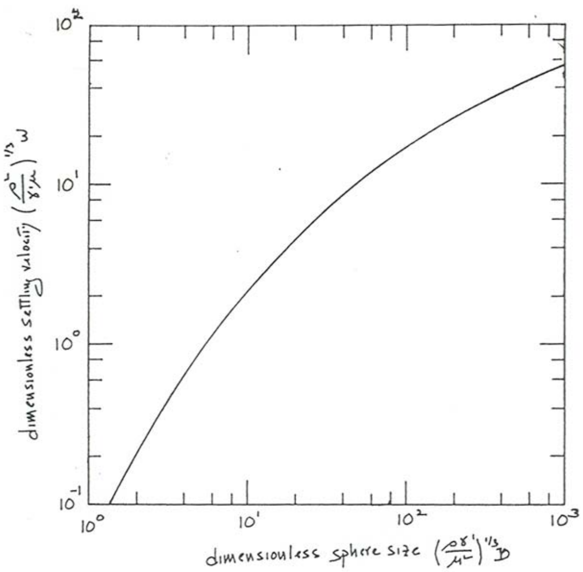

Figure 3-38. Dimensionless settling velocity vs. dimensionless sphere size for settling of a sphere in a vessel of still liquid.

92 Recall from Chapter 2 that if you have a set of dimensionless variables for a problem you can multiply or divide any one of them by any others in the set to get a new variable to replace the old one. To get a dimensionless variable with w but not D, invert the left-hand variable in Equation 3.23 and multiply the result by the right-hand variable:

$$\left(\frac{\rho w^2}{\gamma' D}\right)\left(\frac{\rho w D}{\mu}\right) = \frac{\rho^2 w^3}{\gamma' \mu} \tag{3.24}$$

And to get a dimensionless variable with D but not w, square the right-hand variable in Equation 3.23 and multiply it by the left-hand variable:

$$\left(\frac{\rho w D}{\mu}\right)^2\left(\frac{\gamma' D}{\rho w^2}\right) = \frac{\rho \gamma' D^3}{\mu^2} \tag{3.25}$$

It is convenient, but not necessary, to take these two variables to the one-third power, so that w and D appear to the first power; $w(\rho^2/\gamma'\mu)^{1/3}$ can be viewed as a dimensionless settling velocity, and $D(\rho\gamma'/\mu^2)^{1/3}$ as a dimensionless sphere diameter. Because these two new variables are equivalent to C_D and Re, the functional relationship for C_D vs. Re can just as well be written

$$\left(\frac{\rho^2}{\gamma'\mu}\right)^{1/3} w = f\left(\frac{\rho\gamma'}{\mu^2}\right)^{1/3} D \tag{3.26}$$

The usefulness of Equation 3.26 is that the settling velocity appears only on the left side and the sphere diameter appears only on the right side.

93 It is now a simple matter to find w for a given fluid, sphere size, and submerged specific weight by use of Figure 3-38, which is a plot of dimensionless setting velocity vs. dimensionless sphere diameter. This curve is obtained directly from that in Figure 3-37; you can imagine taking the original data points and forming the new dimensionless variables rather than the old ones to plot the curve in the new coordinate axes of Figure 3-38. This emphasizes that these two curves are equivalent because they are based on the same set of experimental data.

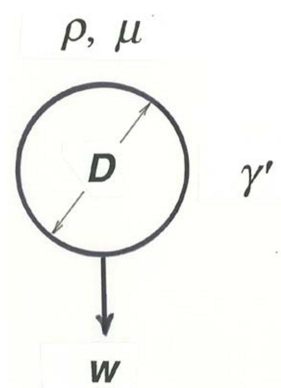

Figure 3-39. Definition sketch for dimensional analysis of a sphere settling through a still fluid.

94 If you are unsatisfied by the roundabout way of arriving at the functional relationship expressed in Equation 3.24, you might consider making a fresh start on dimensional analysis of the problem of settling of a sphere through a still fluid at terminal velocity (Figure 3-39). Settling velocity w, the dependent variable, must depend on fluid density ρ, fluid viscosity μ, sphere diameter D, and submerged weight per unit volume γ' of the sphere. Each of these must be included for the reasons given in Chapter 2. As before, acceleration of gravity and sphere density do not have to appear separately in the list of variables because they are important only by virtue of their combined effect on γ'. The five variables w, ρ, μ, D, and γ' should then combine into two dimensionless variables. You can conveniently arrange for one to contain w but not D and the other to contain D but not w by using the other three variables as the "repeating" variables. You might verify for yourself that this procedure leads to the two dimensionless variables in Equation 3.26.

Settling at Low Reynolds Numbers

95 Remember from Chapter 2 that if the Reynolds number based on sphere diameter and relative flow velocity is less than about one, the drag force on the sphere is given exactly by Stokes' Law, $F_D = 3\pi\mu UD$. This holds in particular for spheres settling under their own weight. Using Stokes' Law it is easy to develop a useful formula for the settling velocity of spheres that is valid in the Stokes range (Re < 1). Write an equation that balances the submerged weight of the sphere, $(\pi D^3/6)\gamma'$, by the drag force, given by Stokes' Law, $3\pi D\mu w$, where I have used the settling velocity w as the relative velocity of the fluid and the sphere. Solving for w,

$$w = \frac{1}{18}\frac{\gamma' D^2}{\mu} \tag{3.27}$$

This equation is widely cited and widely used in books and papers on settling of spheres and other bodies, like sediment particles, that have the approximate shape of a sphere, but keep firmly in mind that it applies *only* in the Stokes range of settling Reynolds numbers $\rho wD/\mu$.

The Effect of Turbulence on Particle Settling

96 One clear effect of turbulence on particle setting is the possibility of dispersion. Think about arranging an experiment in which a batch or continuing supply of sediment particles is introduced at the free surface of a channel flow. If the flow is laminar, the particles (if they are identical in size, shape, and density) are identical also in their settling behavior as well, and they all land on the bottom boundary of the flow at the same point,. If the flow is turbulent, however, the settling particles land on the bottom over a wide streamwise range of points, the obvious reason being that each particles traverses (or, perhaps more accurately, finds itself within the domain of) a different set of eddies during its descent, and so it experiences a different set of local fluid velocities. Both the vertical and the horizontal components of fluid velocities within eddies act to spread out, or disperse, the particle trajectories around the overall average trajectory.

97 It might seem intuitively reasonable to you to assume that the average settling time of the particles in the turbulent flow is the same as the single, well-defined settling time in the laminar flow, because the particles in the turbulent flow are, when viewed on the scale of the particles themselves, just settling through surrounding fluid that is the same as in the laminar case, and the "ups" should balance out the "downs", on average. That cannot be quite true, however, if only for the reason that the drag coefficient of a sphere moving through a fluid is non-negligibly affected by accelerations and decelerations of the fluid that is moving relative to the particle, and that is exactly what is happening when the particle is in a turbulent flow field.

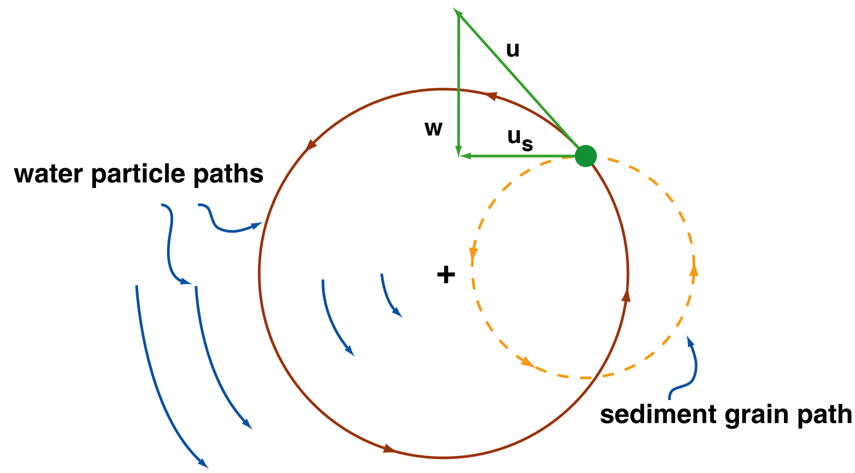

Figure by MIT OpenCourseWare.

Figure 3-40. Path of a sediment particle moving in a fluid vortex. (Figure from G.V. Middleton.)

98 But there is another effect, and one that cannot be ignored, that almost certainly lies outside of your intuition: under certain conditions, a settling particle that finds itself within a rotating eddy tends to become trapped within that eddy! This effect has been studied by Tooby et al. (1977) and Nielsen (1984). In an ideal vortex, rotating about a horizontal axis, a particle within the rising limb describes a circular orbit, which ideally would be closed and would not exhibit any net downward motion (Figure 3-40). In Figure 3-40, the tangential velocity of the fluid, u, is proportional to the distance from the center of the vortex. Simple vector addition of u and w, the settling velocity of the particle, products a circular trajectory of the sediment particle, with no net settling. In the experiment by Tooby et al. (1977), particles tended to spiral slowly outwards, so that they would ultimately diffuse out of the vortex, but sufficiently fine sediment would be trapped in vortices and would move with the vortex until it dissipated. The mechanism is not very sensitive to the size of the trapped particles, and it should tend to produce less vertical segregation by size than the classical diffusional theory predicts.

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