LECTURE # 14

- GYROSCOPES

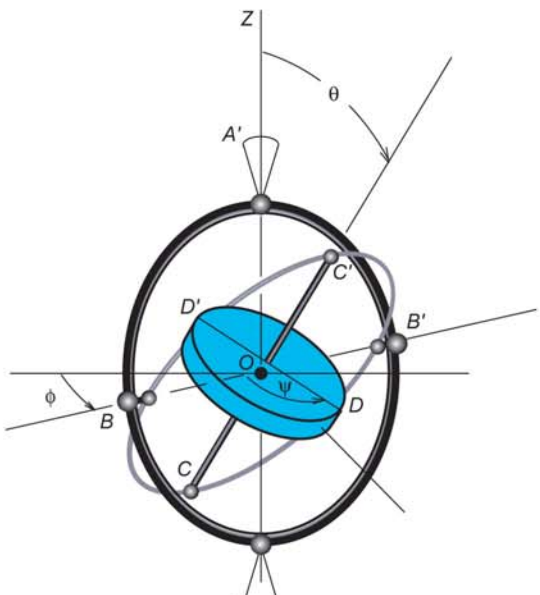
- MODIFIED TRANSPORT THEOREM
- PRECESSION

GYROSCOPES

- UPTO NOW, HAVE CONSIDERED PROBLEMS RELEVANT TO THE RIGID BODY DYNAMICS THAT ARE IMPORTANT TO AEROSPACE VEHICLES
  - USED A BODY FRAME THAT ROTATES WITH THE VEHICLE

- ANOTHER IMPORTANT CLASS OF PROBLEMS FOR BODIES SUCH AS GYROSCOPES
  - ROTOR WITH HIGH SPIN RATE
  - ESSENTIALLY MASSLESS FRAME (CARDAN)
  - MASS CENTER FIXED, BUT ROTOR CAN ASSUME ANY ORIENTATION.

- NATURAL IN THIS CASE TO USE A ROTATING SET OF COORDINATES ATTACHED TO THE INNER GIMBAL. "G"



⇒ NOW FRAME OF REFERENCE NOT ACTUALLY ATTACHED TO THE BODY (ROTOR)

⇒ NEED TO MODIFY $\vec{\omega}$ USED IN TRANSPORT THM.

- RECALL THAT WE HAD $\vec{M} = \dot{\vec{H}}^{I}$

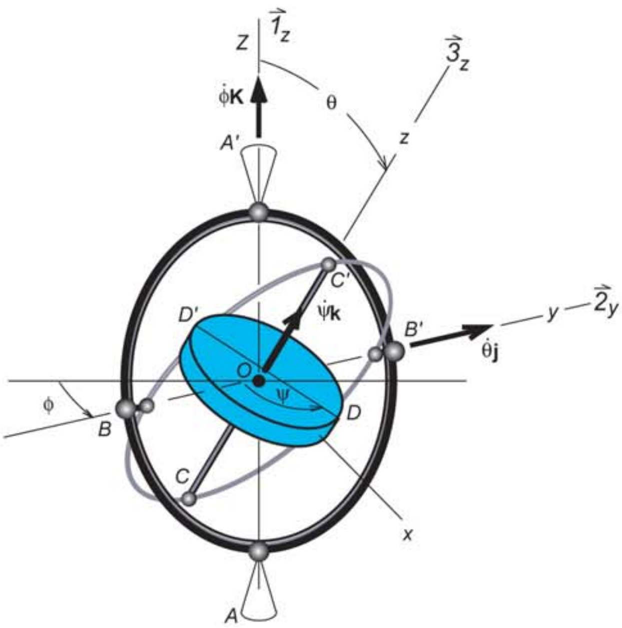
  AND WE SAID THAT $\dot{\vec{H}}^{I} = \dot{\vec{H}}^{B} + \vec{\omega} \times \vec{H}$

  WITH $\vec{\omega}$ BEING THE ABSOLUTE ANGULAR VELOCITY OF THE BODY.

  BY ASSUMPTION, WE ALSO HAD $\vec{H} = \vec{\vec{I}} \cdot \vec{\omega}$

- IN THE GYRO CASE, WE SEE THAT THERE ARE TWO ANGULAR VELOCITIES OF INTEREST.

  - IN TERMS OF THE EULER ANGLE RATES WE HAVE:



TOTAL ANG. VELOCITY:

$$\vec{\omega} = \dot{\phi}\,\vec{1}_z + \dot{\theta}\,\vec{2}_y + \dot{\psi}\,\vec{3}_z$$

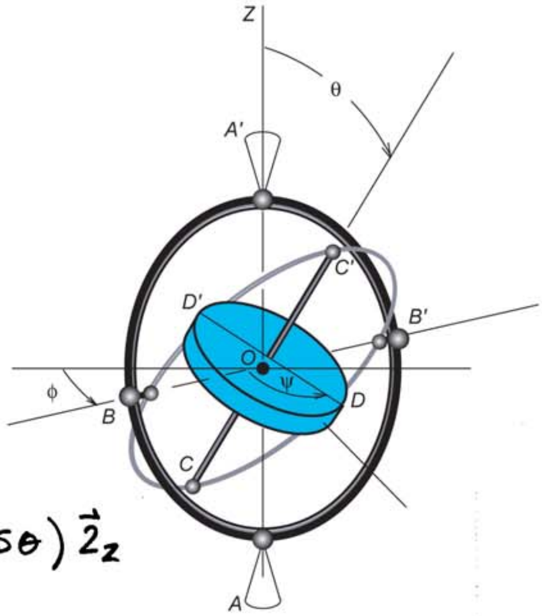
ANGULAR VELOCITY OF INNER GIMBAL AXES:

$$\vec{\Omega} = \dot{\phi}\,\vec{1}_z + \dot{\theta}\,\vec{2}_y$$

- TRANSPORT THEOREM IN THIS CASE:

$$\vec{M} = \dot{\vec{H}}^{I} = \dot{\vec{H}}^{G} + \vec{\Omega} \times \vec{H} \qquad \vec{H} = \vec{\vec{I}} \cdot \vec{\omega}$$

- TO PROCEED, MUST WRITE $\vec{\omega}$ AND $\vec{\Omega}$ USING THE BASIS VECTORS OF THE INNER GIMBAL FRAME.



$$\vec{1}_z = -\sin\theta\ \vec{2}_x + \cos\theta\ \vec{2}_z$$

$$\vec{3}_z = \vec{2}_z$$

- SO

$$\vec{\omega} = \dot{\phi}\left(-\sin\theta\ \vec{2}_x + \cos\theta\ \vec{2}_z\right) + \dot{\theta}\ \vec{2}_y + \dot{\psi}\ \vec{2}_z$$

$$= -\dot{\phi}\sin\theta\ \vec{2}_x + \dot{\theta}\ \vec{2}_y + \left(\dot{\psi} + \dot{\phi}\cos\theta\right)\vec{2}_z$$

AND

$$\vec{\Omega} = -\dot{\phi}\sin\theta\ \vec{2}_x + \dot{\theta}\ \vec{2}_y + \dot{\phi}\cos\theta\ \vec{2}_z \qquad \left(\text{NO } \dot{\psi}\,\vec{3}_z\right)$$

- ANGULAR MOMENTUM - IGNORE MASS OF GIMBALS AND ASSUME I_s ~ MOMENT OF INERTIA ABOUT SPIN AXIS OF ROTOR

  I_t ~ ABOUT TRANSVERSE AXIS.

$$H_G = I_G\, \omega_G = \begin{bmatrix} I_t & & 0 \\ & I_t & \\ 0 & & I_s \end{bmatrix} \begin{bmatrix} -\dot{\phi}\sin\theta \\ \dot{\theta} \\ \dot{\psi} + \dot{\phi}\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} -I_t\,\dot{\phi}\sin\theta \\ I_t\,\dot{\theta} \\ I_s\left(\dot{\psi} + \dot{\phi}\cos\theta\right) \end{bmatrix}$$

NOTE: ROTOR IS MOVING WRT "G" FRAME, BUT DUE TO SYMMETRY, I_G IS CONSTANT.

- SO, IN TERMS OF THE FRAME ATTACHED TO THE INNER GIMBAL:

$$M_G = \dot{H}_G^G + \Omega_G^\times H_G$$

$$\dot{H}_G^G = \begin{bmatrix} -I_t\left(\ddot{\phi}\sin\theta + \dot{\phi}\dot{\theta}\cos\theta\right) \\ I_t\ddot{\theta} \\ I_s\left(\ddot{\psi} + \ddot{\phi}\cos\theta - \dot{\phi}\dot{\theta}\sin\theta\right) \end{bmatrix}$$

$$\Omega_G^\times H_G = \begin{bmatrix} 0 & -\dot{\phi}\cos\theta & \dot{\theta} \\ \dot{\phi}\cos\theta & 0 & \dot{\phi}\sin\theta \\ -\dot{\theta} & -\dot{\phi}\sin\theta & 0 \end{bmatrix}\begin{bmatrix} -I_t\dot{\phi}\sin\theta \\ I_t\dot{\theta} \\ I_s\left(\dot{\psi} + \dot{\phi}\cos\theta\right) \end{bmatrix}$$

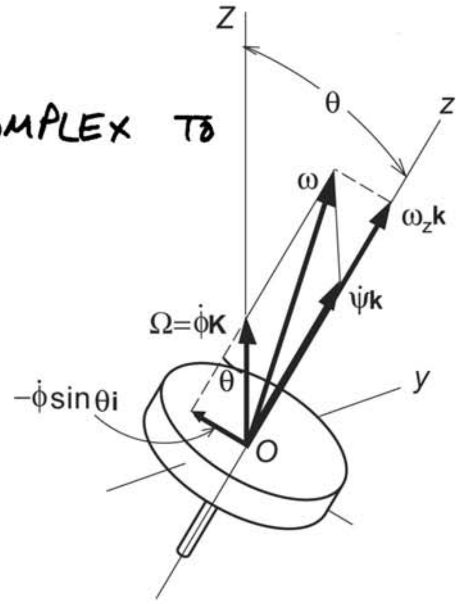
$$M_x = -I_t\left(\ddot{\phi}\sin\theta + 2\dot{\phi}\dot{\theta}\cos\theta\right) + I_s\dot{\theta}\left(\dot{\psi} + \dot{\phi}\cos\theta\right)$$

$$M_y = I_t\left(\ddot{\theta} - \dot{\phi}^2\cos\theta\sin\theta\right) + I_s\dot{\phi}\sin\theta\left(\dot{\psi} + \dot{\phi}\cos\theta\right)$$

$$M_z = I_s\left(\ddot{\psi} + \ddot{\phi}\cos\theta - \dot{\phi}\dot{\theta}\sin\theta\right)$$

- NONLINEAR, 2^{ND} ORDER, FULLY COUPLED
  - ⇒ HARD TO SOLVE FOR ϕ, θ, ψ GIVEN M_x, M_y, M_z
  - ⇒ LOOK AT SOME SPECIAL CASES.

STEADY PRECESSION



- EQUATIONS OF MOTION FAR TOO COMPLEX TO SOLVE IN GENERAL.
  - INTERESTING SUB-PROBLEM.

- ASSUME:
  1) ANGLE θ (NUTATION) CONSTANT.
  2) ANGLE RATE $\dot\phi$ (PRECESSION RATE) CONSTANT
  3) ROTOR SPIN $\dot\psi$ CONSTANT.

$$\left.\begin{array}{l} 1) \Rightarrow \theta = C_1 , \quad \dot\theta = \ddot\theta = 0 \\ 2) \Rightarrow \dot\phi = C_2 , \quad \ddot\phi = 0 \\ 3) \Rightarrow \dot\psi = C_3 , \quad \ddot\psi = 0 \end{array}\right\} \quad \begin{array}{l} M_x = 0 \\ M_z = 0 \end{array}$$

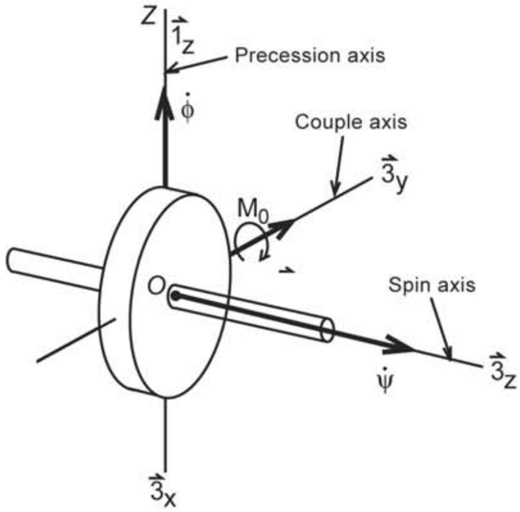
AND

$$M_y = I_t\left(-\dot\phi^2 \cos\theta \sin\theta\right) + I_s \dot\phi \sin\theta(\dot\psi + \dot\phi\cos\theta)$$

$$= \dot\phi \sin\theta \left[I_s(\dot\psi + \dot\phi\cos\theta) - I_t \dot\phi \cos\theta \right]$$

- FOR A GIVEN $\theta, M_y, I_s, I_t, \dot\psi$ WE CAN PREDICT THE PRECESSION RATE!
  - WHAT IF $\theta = 90°$?

- IN THE CASE $\theta = 90^\circ$, THE ROTOR SPIN AXIS IS IN THE HORIZONTAL PLANE



$$\Rightarrow \quad M_Y = I_s \dot{\phi} \dot{\psi}$$

$$\dot{\phi} = \frac{M_Y}{I_s \dot{\psi}}$$

- NOW EXPLICIT THAT IF WE APPLY A MOMENT TO A GYROSCOPE ABOUT AN AXIS PERPENDICULAR TO ITS AXIS OF SPIN, THE GYROSCOPE WILL PRECESS ABOUT AN AXIS PERPENDICULAR TO <u>BOTH</u> THE SPIN AXIS AND THE MOMENT AXIS.

$$\left.\begin{array}{l} \text{– TORQUE ABOUT } \vec{2}_Y \text{ AXIS} \\ \text{– SPIN ABOUT } \vec{2}_Z \text{ AXIS} \end{array}\right\} \quad \text{PRECESS ABOUT } \vec{2}_X \text{ (VERTICAL)}$$

- <u>DIRECTION OF PRECESSION:</u>
  CAUSES POSITIVE END OF SPIN AXIS TO ROTATE TOWARDS POSITIVE END OF MOMENT AXIS.

OBSERVATIONS :

1) SINCE $\dot{\phi} = \dfrac{M_Y}{I_s \dot{\psi}}$

$\Rightarrow$ FOR A GIVEN EXTERNAL MOMENT, THE GREATER THE SPIN ($\dot{\psi}$), THE SLOWER THE PRECESSION ($\dot{\phi}$)

"SPIN STABILIZATION"

2) BECAUSE OF THE RELATIVELY LARGE COUPLES REQUIRED TO CHANGE THE ORIENTATION OF THE SPIN AXLE, GYROSCOPES CAN BE USED TO STABILIZE TORPEDOES AND SHIPS

3) SEE EXAMPLES.